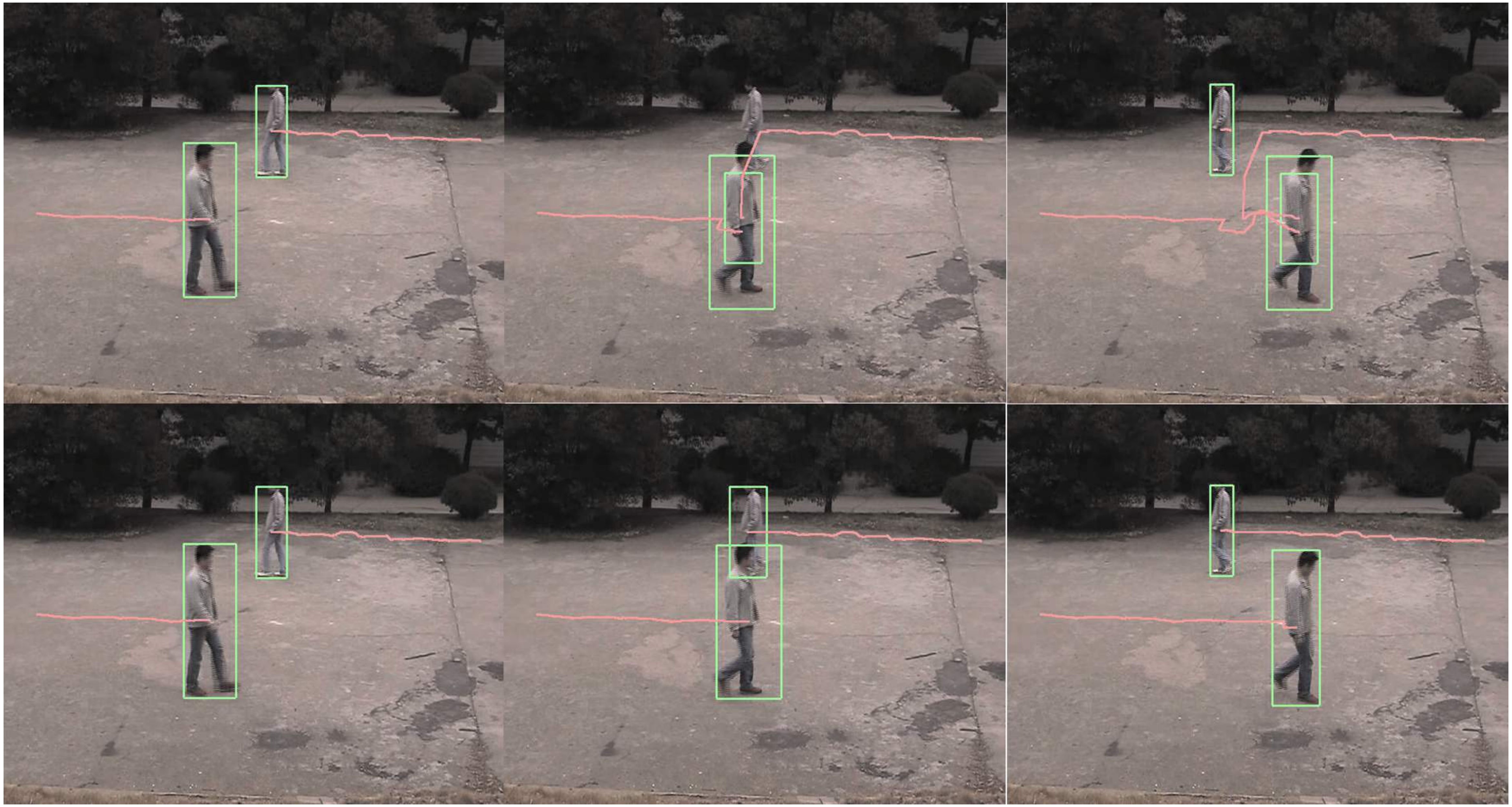
Alignment and tracking



16-385 Computer Vision
Fall 2020, Lecture 24 & 25

http://16385.courses.cs.cmu.edu/

Overview of today's lecture

- Motion magnification using optical flow.
- Image alignment.
- Lucas-Kanade alignment.
- Baker-Matthews alignment.
- Inverse alignment.
- KLT tracking.
- Mean-shift tracking.
- Modern trackers.

Slide credits

Most of these slides were adapted from:

- Kris Kitani (16-385, Spring 2017).

Motion magnification using optical flow

How would you achieve this effect?

original

motion-magnified

- Compute optical flow from frame to frame.
- Magnify optical flow velocities.
- Appropriately warp image intensities.

How would you achieve this effect?

naïvely motion-magnified

motion-magnified

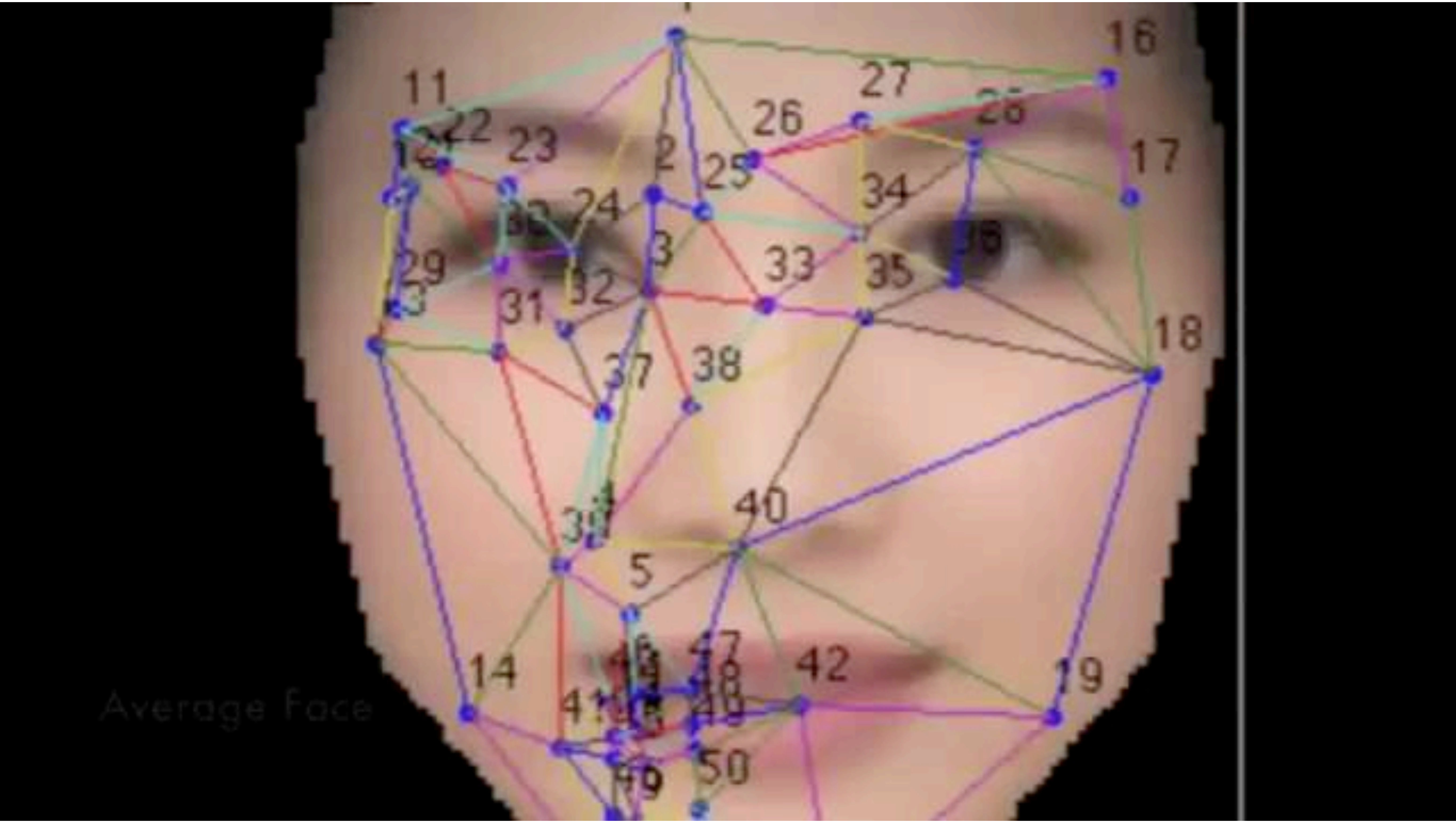
- Compute optical flow from frame to frame.
- Magnify optical flow velocities.
- Appropriately warp image intensities.

In practice, many additional steps are required for a good result.

Some more examples

Some more examples

Image alignment


Average Face
1
2
3
5
11
14
16
17
18
19
22
23
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25
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27
28
29
31
32
33
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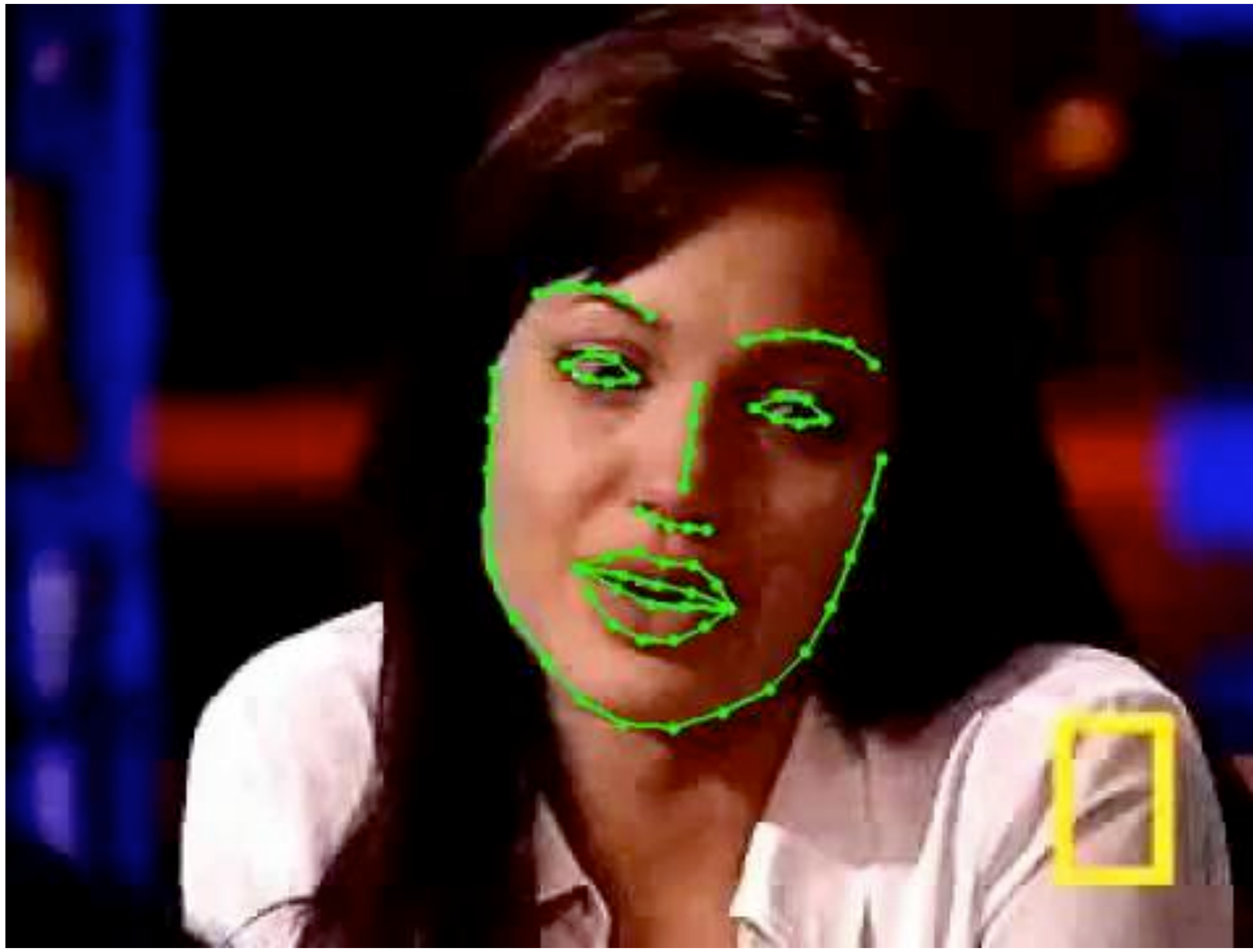
50



IntraFace

http://www.humansensing.cs.cmu.edu/intraface/


31 fps

How can I find  in the image?

Idea #1: Template Matching

Slow, combinatory, global solution

Idea #2: Pyramid Template Matching

Faster, combinatory, locally optimal

Idea #3: Model refinement

(when you have a good initial solution)

Fastest, locally optimal

Some notation before we get into the math…

2D image transformation

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p})$$

2D image coordinate

$$\boldsymbol{x} = \left[\begin{array}{c} x \\ y \end{array}\right]$$

Parameters of the transformation

$$\boldsymbol{p} = \{p_1, \ldots, p_N\}$$

Warped image

$$I(\boldsymbol{x}') = I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$$

Pixel value at a coordinate

Translation

Affine

Some notation before we get into the math…

2D image transformation

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Pixel value at a coordinate

Translation

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) = \left[\begin{array}{c} x + p_1 \\ y + p_2 \end{array}\right]$$

$$= \underbrace{\left[\begin{array}{ccc} 1 & 0 & p_1 \\ 0 & 1 & p_2 \end{array}\right]}_{\text{transform}} \underbrace{\left[\begin{array}{c} x \\ y \\ 1 \end{array}\right]}_{\text{coordinate}}$$

Affine

Some notation before we get into the math…

2D image transformation

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p})$$

2D image coordinate

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$$\boldsymbol{p} = \{p_1, \ldots, p_N\}$$

Warped image

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Pixel value at a coordinate

Translation

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) = \begin{bmatrix} x + p_1 \\ y + p_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & p_1 \\ 0 & 1 & p_2 \end{bmatrix}}_{\text{transform}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\text{coordinate}}$$

Affine

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) = \begin{bmatrix} p_1 x + p_2 y + p_3 \\ p_4 x + p_5 y + p_6 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix}}_{\text{affine transform}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\text{coordinate}}$$

can be written in matrix form when linear

affine warp matrix can also be 3x3 when last row is [0 0 1]

$\mathbf{W}(\boldsymbol{x};\boldsymbol{p})$ takes a ________ as input and returns a _______

$\mathbf{W}(\boldsymbol{x};\boldsymbol{p})$ is a function of ____ variables

$\mathbf{W}(\boldsymbol{x};\boldsymbol{p})$ returns a ______ of dimension ___ x ___

$\boldsymbol{p} = \{p_1, \ldots, p_N\}$ where N is _____ for an affine model

$I(\boldsymbol{x}') = I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$ this warp changes pixel values?

Image alignment

(problem definition)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

warped image template image

Find the warp parameters **p** such that the SSD is minimized

Find the warp parameters **p** such that the SSD is minimized

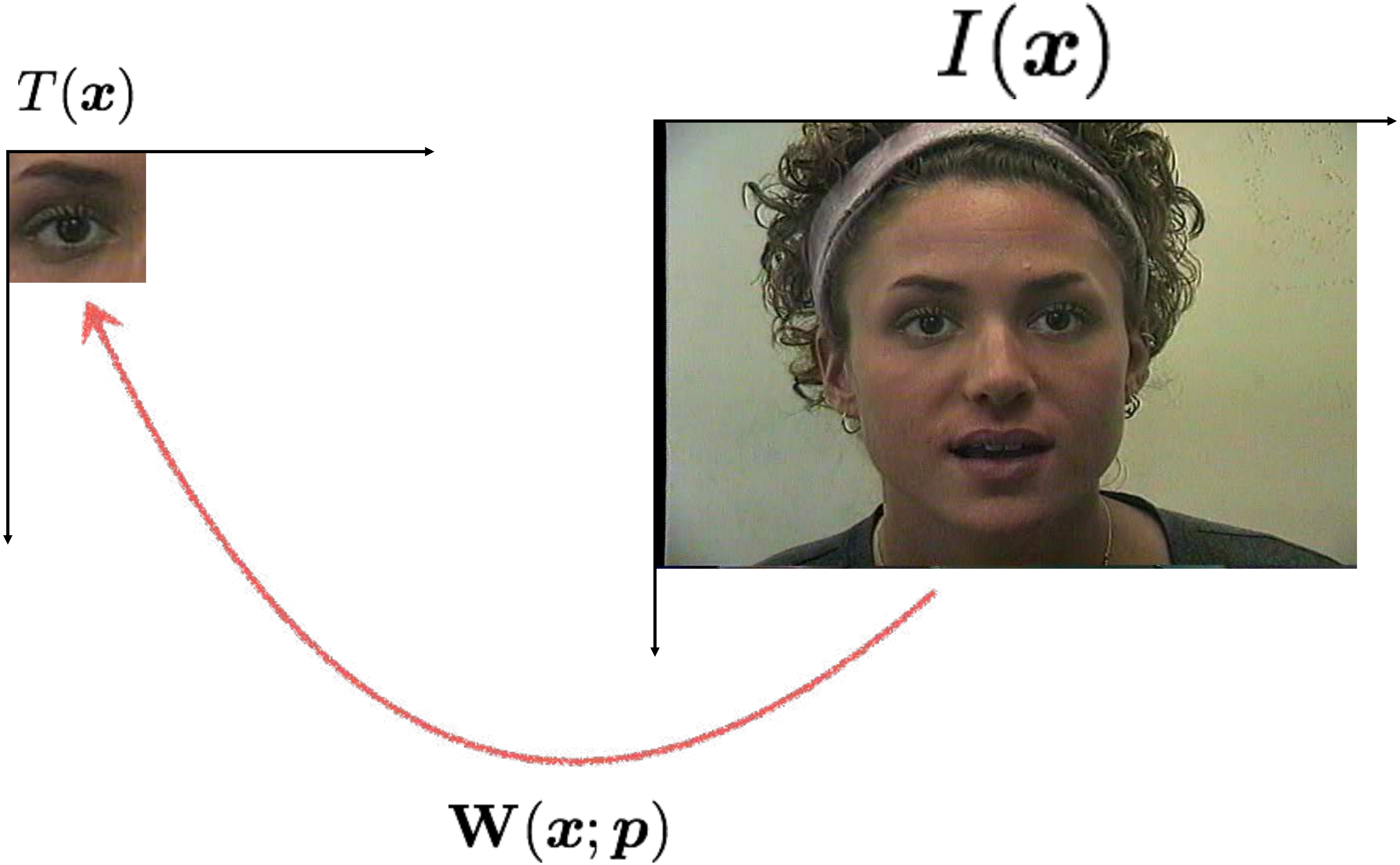


Image alignment

(problem definition)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

warped image

template image

Find the warp parameters **p** such that the SSD is minimized

How could you find a solution to this problem?

This is a non-linear (quadratic) function of a non-parametric function!

(Function $\boldsymbol{I}$ is non-parametric)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Hard to optimize

What can you do to make it easier to solve?

This is a non-linear (quadratic) function of a non-parametric function!

(Function I is non-parametric)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Hard to optimize

What can you do to make it easier to solve?

assume good initialization,
linearized objective and update incrementally

Lucas-Kanade alignment

(pretty strong assumption)

If you have a good initial guess ***p***...

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

can be written as ...

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p} + \Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

(a small incremental adjustment)
(this is what we are solving for now)

This is **still** a non-linear (quadratic) function of a non-parametric function!

(Function $\boldsymbol{I}$ is non-parametric)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

How can we linearize the function $\boldsymbol{I}$ *for a really small perturbation of* ***p****?*

This is **still** a non-linear (quadratic) function of a non-parametric function!

(Function $\boldsymbol{I}$ is non-parametric)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

*How can we linearize the function $\boldsymbol{I}$ for a really small perturbation of **p**?*

Taylor series approximation!

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Multivariable Taylor Series Expansion
(First order approximation)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Multivariable Taylor Series Expansion
(First order approximation)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Recall: $\boldsymbol{x}' = \mathbf{W}(\boldsymbol{x};\boldsymbol{p})$

$$I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) \approx I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \frac{\partial I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))}{\partial \boldsymbol{p}}\Delta\boldsymbol{p}$$

chain rule
$$= I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \frac{\partial I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))}{\partial \boldsymbol{x}'}\frac{\partial \mathbf{W}(\boldsymbol{x};\boldsymbol{p})}{\partial \boldsymbol{p}}\Delta\boldsymbol{p}$$

short-hand
$$= I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\Delta\boldsymbol{p}$$

short-hand

$$\sum_{\boldsymbol{x}} [\underline{I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p}))} - T(\boldsymbol{x})]^2$$

Multivariable Taylor Series Expansion
(First order approximation)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linear approximation

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

What are the unknowns here?

$$\sum_{\boldsymbol{x}} [\underline{I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p}))} - T(\boldsymbol{x})]^2$$

Multivariable Taylor Series Expansion
(First order approximation)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linear approximation

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Now, the function is a linear function of the unknowns

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

$\boldsymbol{x}$ is a __________ of dimension ___ x ___

output of $\mathbf{W}$ is a __________ of dimension ___ x ___

$\boldsymbol{p}$ is a ___________ of dimension ___ x ___

$I(\cdot)$ is a function of _____ variables

The Jacobian $\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}$

(A matrix of partial derivatives)

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} W_x(x,y) \\ W_y(x,y) \end{bmatrix}$$

$$\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} = \begin{bmatrix} \frac{\partial W_x}{\partial p_1} & \frac{\partial W_x}{\partial p_2} & \cdots & \frac{\partial W_x}{\partial p_N} \\ \frac{\partial W_y}{\partial p_1} & \frac{\partial W_y}{\partial p_2} & \cdots & \frac{\partial W_y}{\partial p_N} \end{bmatrix}$$

Rate of change of the warp

Affine transform

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) = \begin{bmatrix} p_1 x + p_3 y + p_5 \\ p_2 x + p_4 y + p_6 \end{bmatrix}$$

$$\frac{\partial W_x}{\partial p_1} = x \qquad \frac{\partial W_x}{\partial p_2} = 0 \qquad \cdots$$

$$\frac{\partial W_y}{\partial p_1} = 0 \qquad \cdots$$

$$\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} = \begin{bmatrix} x & 0 & y & 0 & 1 & 0 \\ 0 & x & 0 & y & 0 & 1 \end{bmatrix}$$

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

∇I is a __________ of dimension ___ x ___

$\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}$ is a __________ of dimension ___ x ___

$\Delta \boldsymbol{p}$ is a __________ of dimension ___ x ___

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

pixel coordinate
(2 x 1)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

pixel coordinate
(2 x 1)

image intensity
(scalar)

warp function
(2 x 1)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

pixel coordinate
(2 x 1)

image intensity
(scalar)

warp function
(2 x 1)

warp parameters
(6 for affine)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

pixel coordinate
(2 x 1)

image intensity
(scalar)

warp function
(2 x 1)

warp parameters
(6 for affine)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

image gradient
(1 x 2)

pixel coordinate
(2 x 1)

image intensity
(scalar)

Partial derivatives of warp function
(2 x 6)

warp function
(2 x 1)

warp parameters
(6 for affine)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

image gradient
(1 x 2)

pixel coordinate
(2 x 1)

image intensity
(scalar)

Partial derivatives of warp function
(2 x 6)

warp function
(2 x 1)

warp parameters
(6 for affine)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

image gradient
(1 x 2)

incremental warp
(6 x 1)

pixel coordinate
(2 x 1)

image intensity
(scalar)

Partial derivatives of warp function
(2 x 6)

warp function
(2 x 1)

template image intensity
(scalar)

warp parameters
(6 for affine)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

image gradient
(1 x 2)

incremental warp
(6 x 1)

pixel coordinate
(2 x 1)

image intensity
(scalar)

When you implement this, you will compute everything in parallel and store as matrix … don’t loop over x!

Summary

(of Lucas-Kanade Image Alignment)

Problem:

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

warped image   template image

Difficult non-linear optimization problem

Strategy:

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p} + \Delta \boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

<u>Assume</u> known approximate solution

Solve for increment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Taylor series approximation

Linearize

then solve for $\Delta \boldsymbol{p}$

OK, so how do we solve this?

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Another way to look at it…

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

(moving terms around)

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - \{ T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) \} \right]^2$$

vector of constants

vector of variables

constant

Have you seen this form of optimization problem before?

Another way to look at it…

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - \{ T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) \} \right]^2$$

constant variable constant

Looks like $\mathbf{A}\mathbf{x} - \mathbf{b}$

How do you solve this?

Least squares approximation

$$\hat{x} = \arg\min_x ||Ax - b||^2$$ is solved by $$x = (A^\top A)^{-1} A^\top b$$

Applied to our tasks:

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - \{ T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p})) \} \right]^2$$

is optimized when

$$\Delta \boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}))]$$

after applying

$x = (A^\top A)^{-1} A^\top b$

where $$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

$A^\top A$

Solve:

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

warped image

template image

Difficult non-linear optimization problem

Strategy:

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p} + \Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Assume known approximate solution

Solve for increment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Taylor series approximation

Linearize

Solution:

$$\Delta\boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} \left[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) \right]$$

Solution to least squares approximation

$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

Hessian

This is called…

Gauss-Newton gradient decent non-linear optimization!

Lucas Kanade (Additive alignment)

1. Warp image $I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$
2. Compute error image $[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$
3. Compute gradient $\nabla I(\boldsymbol{x}')$

   x'coordinates of the warped image (gradients of the warped image)
4. Evaluate Jacobian $\dfrac{\partial \mathbf{W}}{\partial \boldsymbol{p}}$
5. Compute Hessian H

   $$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^{\top} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]$$
6. Compute $\Delta \boldsymbol{p}$

   $$\Delta \boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^{\top} [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$$
7. Update parameters $\boldsymbol{p} \leftarrow \boldsymbol{p} + \Delta \boldsymbol{p}$

Just 8 lines of code!

Baker-Matthews alignment

Image Alignment

(start with an initial solution, match the image and template)

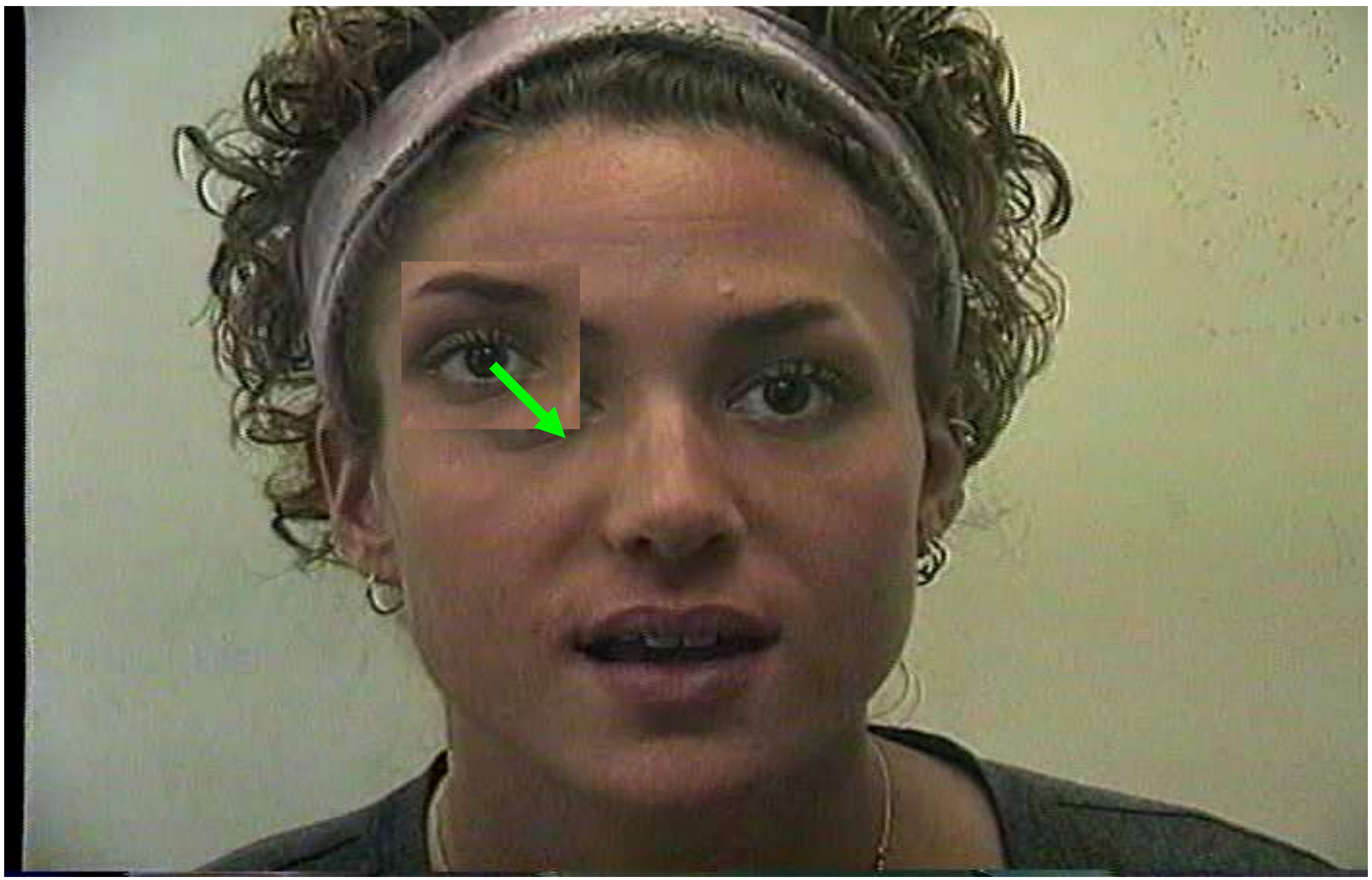


Image Alignment Objective Function

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Given an initial solution…several possible formulations

Additive Alignment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p} + \Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

incremental perturbation of parameters

Image Alignment Objective Function

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Given an initial solution…several possible formulations

Additive Alignment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

incremental perturbation of <u>parameters</u>

Compositional Alignment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x}) \right]^2$$

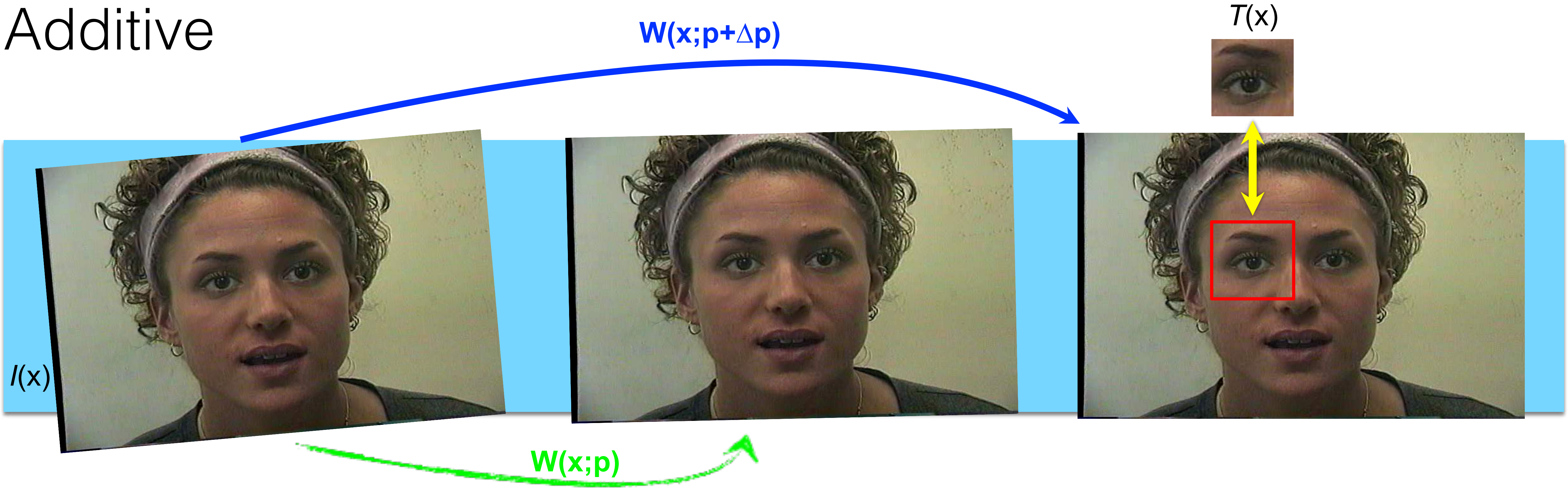
incremental <u>warps</u> of image

Additive strategy

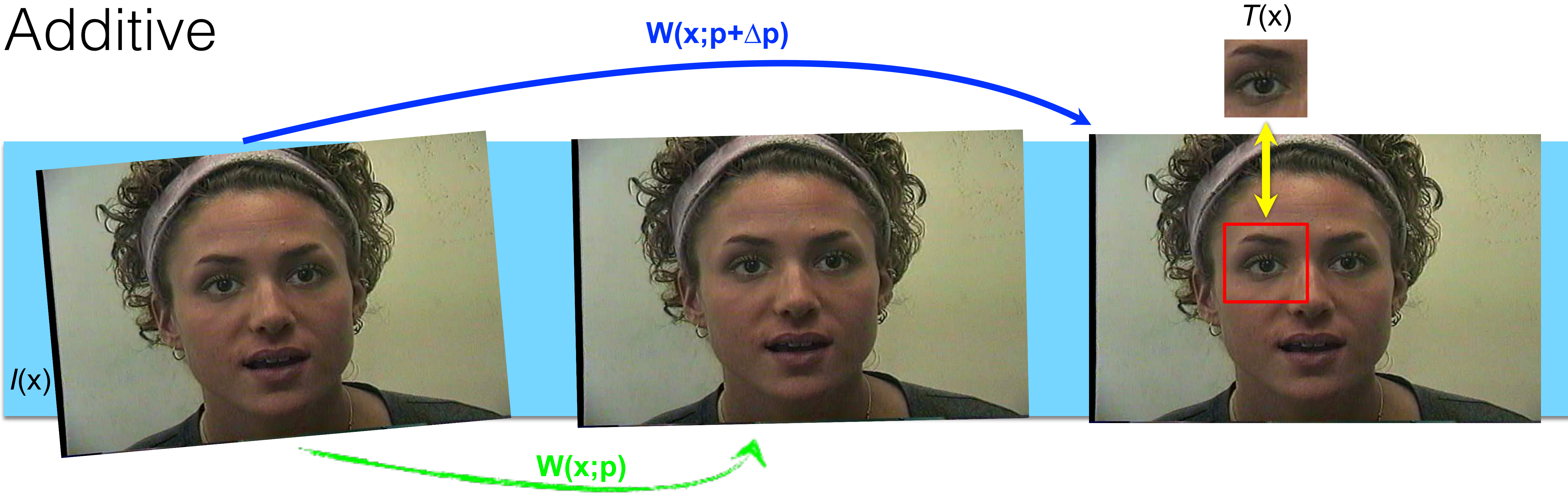
Compositional strategy

Additive

Additive

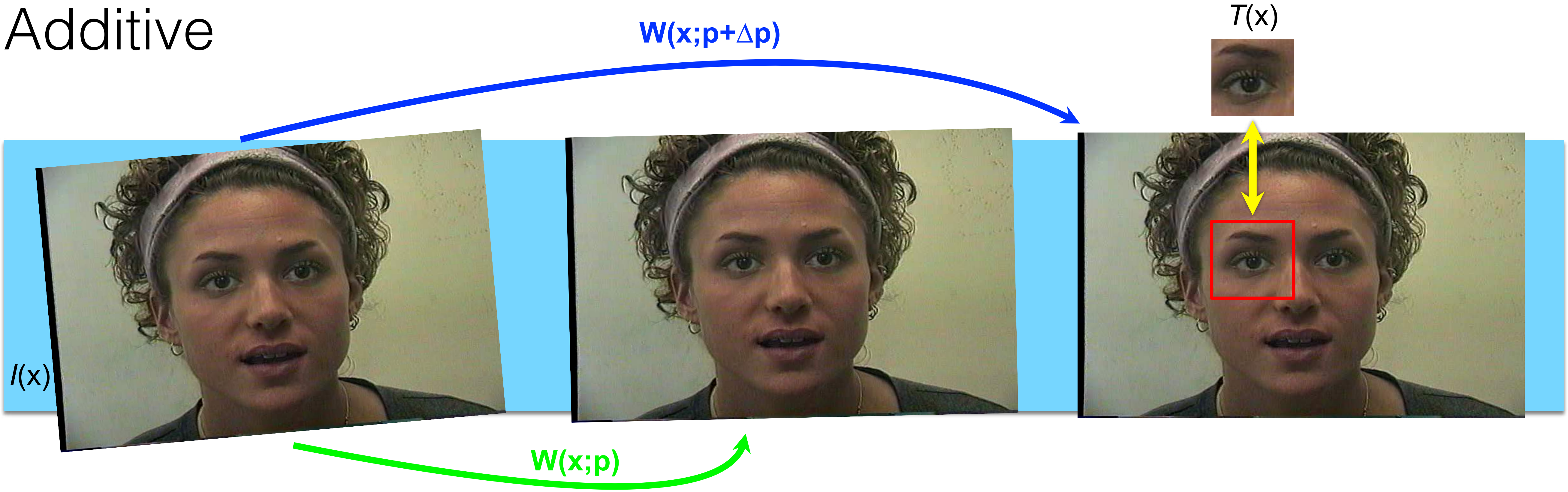


Additive

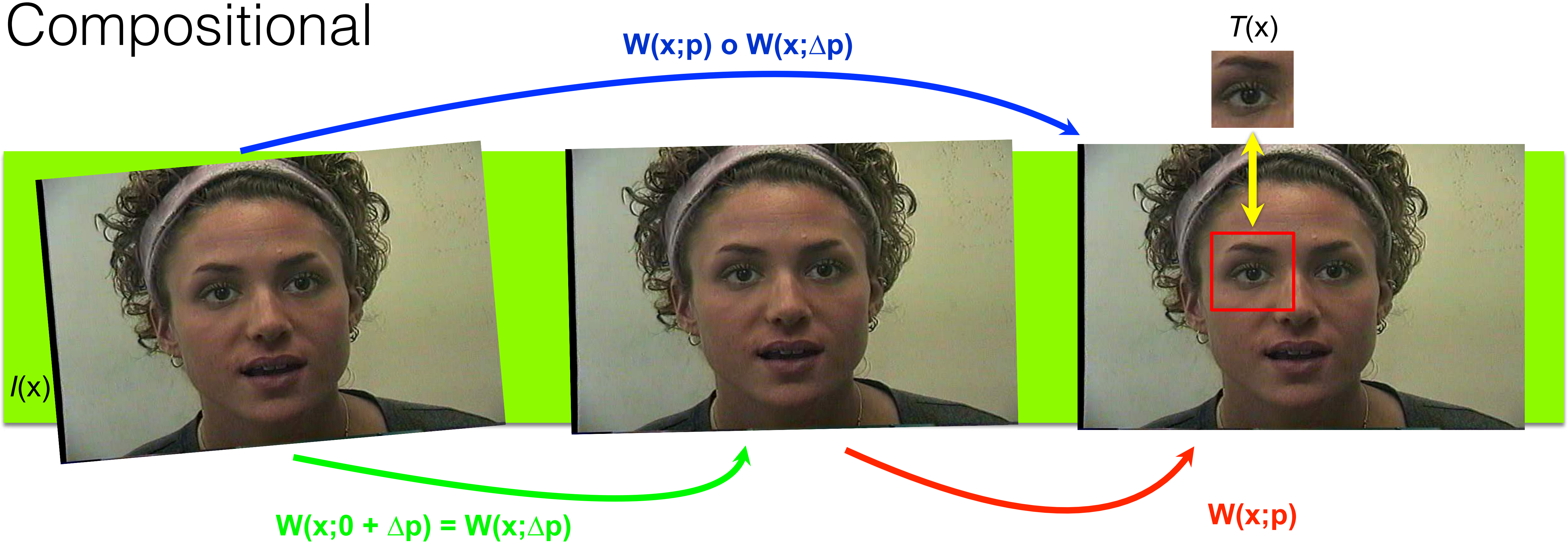


Compositional

Additive



Compositional



Compositional Alignment

Original objective function (SSD)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Assuming an initial solution **p** and a compositional warp increment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x}) \right]^2$$

Compositional Alignment

Original objective function (SSD)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Assuming an initial solution **p** and a compositional warp increment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x}) \right]^2$$

Another way to write the composition

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p}) \equiv \mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ )$$

Identity warp

$$\mathbf{W}(\boldsymbol{x};\mathbf{0})$$

Compositional Alignment

Original objective function (SSD)

$$\min_{\boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Assuming an initial solution **p** and a compositional warp increment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x}) \right]^2$$

Another way to write the composition

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p}) \equiv \mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ )$$

Identity warp

$$\mathbf{W}(\boldsymbol{x};\mathbf{0})$$

Skipping over the derivation…the new update rule is

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \leftarrow \mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p})$$

So what’s so great about this compositional form?

Additive Alignment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

linearized form

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I(\boldsymbol{x}') \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Compositional Alignment

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x}) \right]^2$$

linearized form

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I(\boldsymbol{x}') \frac{\partial \mathbf{W}(\boldsymbol{x};\mathbf{0})}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Additive Alignment

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

Compositional Alignment

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x})]^2$$

linearized form

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I(\boldsymbol{x}') \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Jacobian of W(x;p)

linearized form

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I(\boldsymbol{x}') \frac{\partial \mathbf{W}(\boldsymbol{x};\mathbf{0})}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

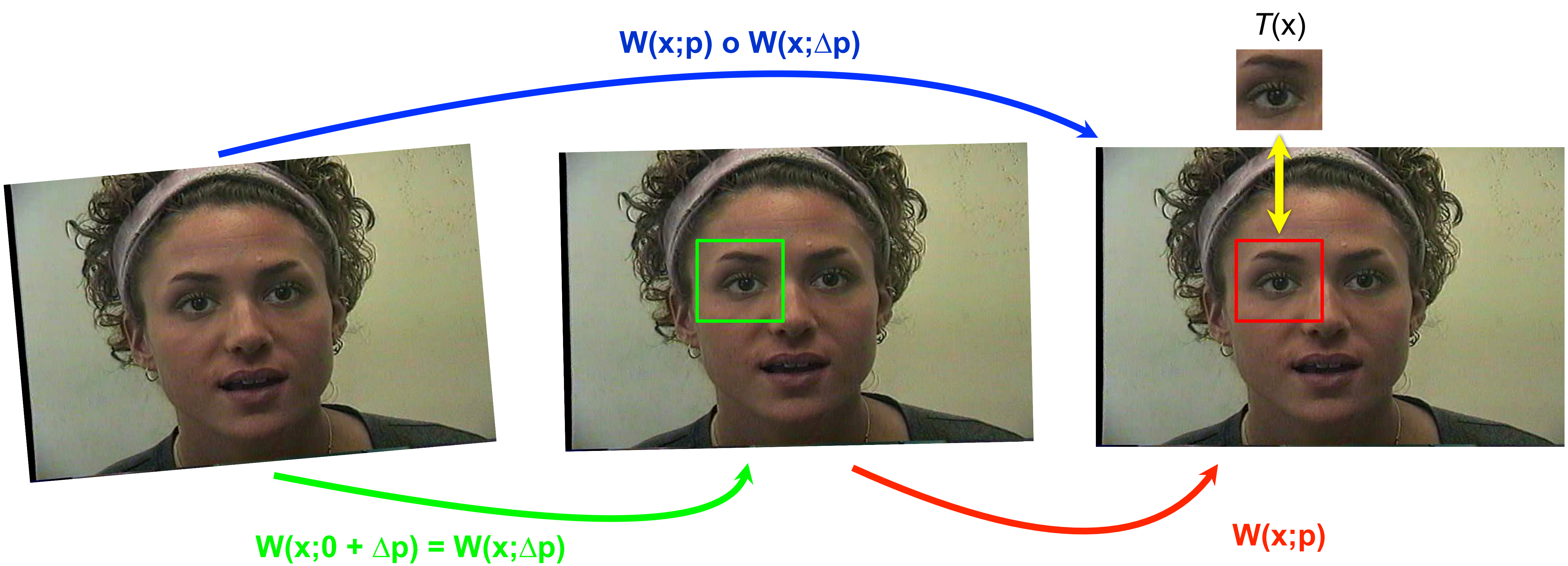
Jacobian of W(x;0)

The Jacobian is constant.
Jacobian can be precomputed!

Compositional Image Alignment

Minimize

$$\sum_{\mathbf{x}} \left[I(\mathbf{W}(\mathbf{W}(\mathbf{x};\Delta\mathbf{p});\mathbf{p})) - T(\mathbf{x}) \right]^2 \approx \sum_{\mathbf{x}} \left[I(\mathbf{W}(\mathbf{x};\mathbf{p})) + \nabla I(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial \mathbf{p}} \Delta\mathbf{p} - T(\mathbf{x}) \right]^2$$



Jacobian is simple and can be precomputed

Lucas Kanade (Additive alignment)

1. Warp image $I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$
2. Compute error image $[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]^2$
3. Compute gradient $\nabla I(\boldsymbol{x}')$
4. Evaluate Jacobian $\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}$
5. Compute Hessian H
6. Compute $\Delta \boldsymbol{p}$
7. Update parameters $\boldsymbol{p} \leftarrow \boldsymbol{p} + \Delta \boldsymbol{p}$

Shum-Szeliski (Compositional alignment)

1. Warp image $I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$
2. Compute error image $[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]^2$
3. Compute gradient $\nabla I(\boldsymbol{x}')$
4. Evaluate Jacobian $\frac{\partial \mathbf{W}(\boldsymbol{x};\mathbf{0})}{\partial \boldsymbol{p}}$
5. Compute Hessian H
6. Compute $\Delta \boldsymbol{p}$
7. Update parameters $\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \leftarrow \mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p})$

Any other speed up techniques?

Inverse alignment

Why not compute warp updates on the template?

Additive Alignment

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

Compositional Alignment

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x})]^2$$

Why not compute warp updates on the template?

Additive Alignment

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

Compositional Alignment

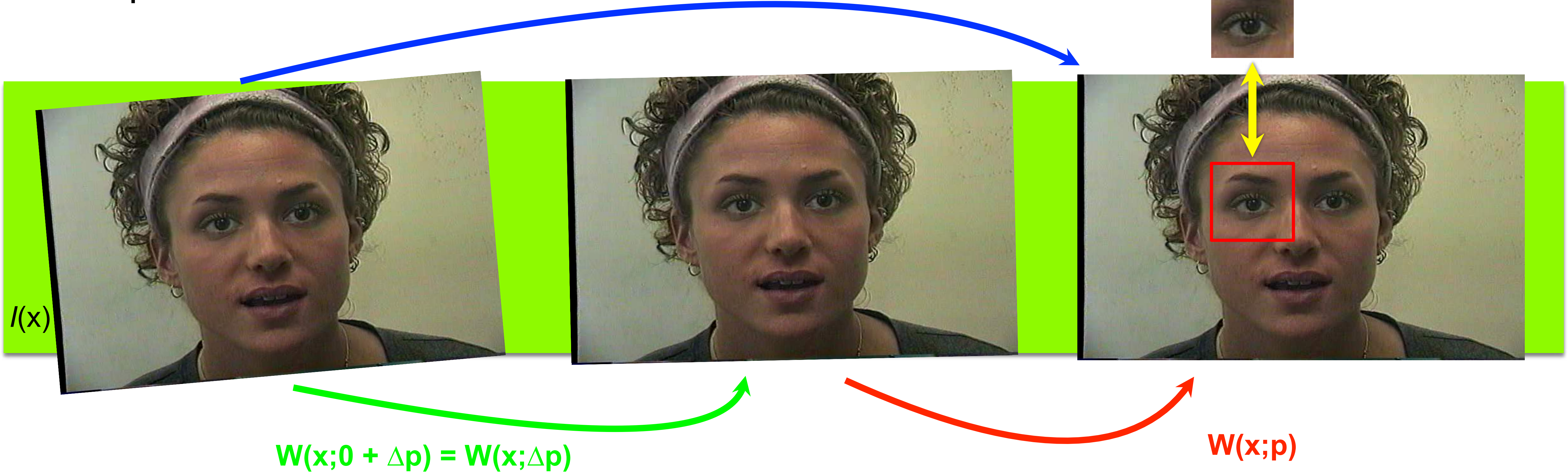
$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p});\boldsymbol{p}\ ) - T(\boldsymbol{x})]^2$$

What happens if you let the template be warped too?

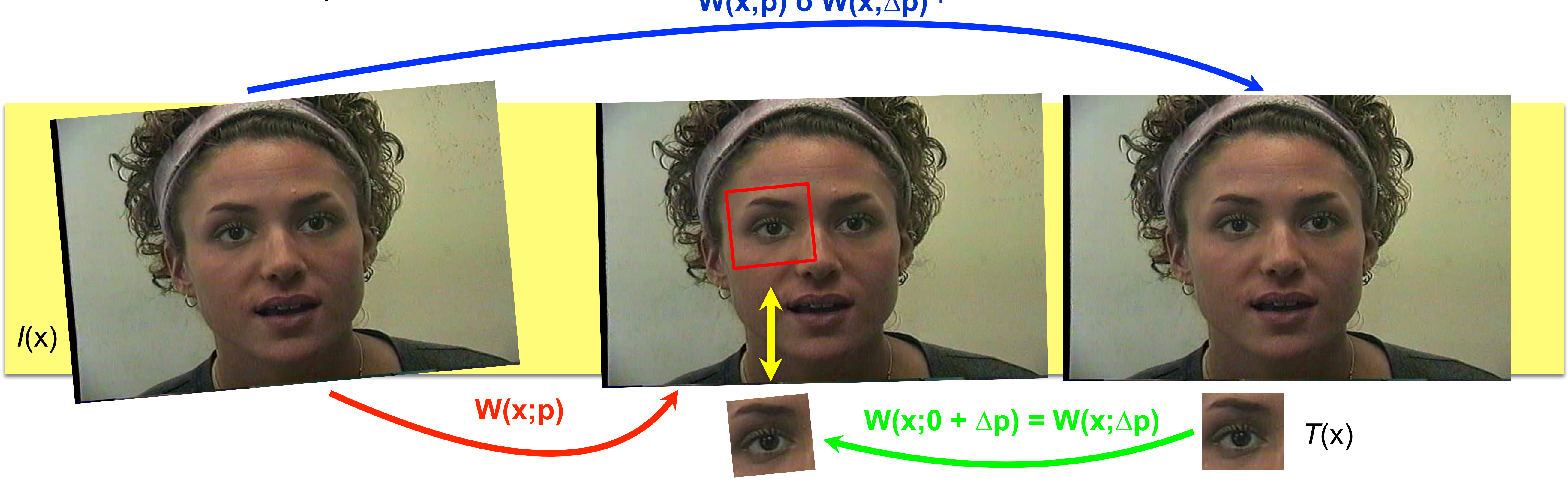
Inverse Compositional Alignment

$$\sum_{\boldsymbol{x}} [T(\mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]^2$$

Compositional



Inverse compositional



Compositional strategy

Inverse Compositional strategy

So what’s so great about this inverse compositional form?

Inverse Compositional Alignment

Minimize

$$\sum_{\boldsymbol{x}} [T(\mathbf{W}(\boldsymbol{x}; \Delta \boldsymbol{p}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}))]^2 \approx \sum \boldsymbol{x} \left[T(\mathbf{W}(\boldsymbol{x}; \mathbf{0})) + \nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}) \right]^2$$

Solution

$$H = \sum_{\boldsymbol{x}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

can be precomputed from template!

$$\Delta \boldsymbol{p} = \sum_{\boldsymbol{x}} H^{-1} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}))]$$

Update

$$\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}) \leftarrow \mathbf{W}(\boldsymbol{x}; \boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x}; \Delta \boldsymbol{p})^{-1}$$

Properties of inverse compositional alignment

Jacobian can be precomputed

It is constant - evaluated at W(x;0)

Gradient of template can be precomputed

It is constant

Hessian can be precomputed

$$\boxed{H = \sum_{\boldsymbol{x}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]}$$

$$\Delta \boldsymbol{p} = \boxed{\sum_{\boldsymbol{x}} H^{-1} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top} [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}))]$$

(main term that needs to be computed)

Warp must be invertible

Lucas Kanade (Additive alignment)

1. Warp image $I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$
2. Compute error image $[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]^2$
3. Compute gradient $\nabla I(\mathbf{W})$
4. Evaluate Jacobian $\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}$
5. Compute Hessian H
6. Compute $\Delta \boldsymbol{p}$
7. Update parameters $\boldsymbol{p} \leftarrow \boldsymbol{p} + \Delta \boldsymbol{p}$

Shum-Szeliski (Compositional alignment)

1. Warp image $I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$
2. Compute error image $[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$
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4. Evaluate Jacobian $\frac{\partial \mathbf{W}(\boldsymbol{x};\mathbf{0})}{\partial \boldsymbol{p}}$
5. Compute Hessian H
6. Compute $\Delta \boldsymbol{p}$
7. Update parameters $\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \leftarrow \mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p})$

Baker-Matthews (Inverse Compositional alignment)

1. Warp image $I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))$

2. Compute error image $[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$

3. Compute gradient $\nabla T(\mathbf{W})$

4. Evaluate Jacobian $\frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}$

5. Compute Hessian H

$$H = \sum_{\boldsymbol{x}} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^\top \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]$$

6. Compute $\Delta \boldsymbol{p}$

$$\Delta \boldsymbol{p} = \sum_{\boldsymbol{x}} H^{-1} \left[\nabla T \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^\top [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$$

7. Update parameters $\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \leftarrow \mathbf{W}(\boldsymbol{x};\boldsymbol{p}) \circ \mathbf{W}(\boldsymbol{x};\Delta\boldsymbol{p})^{-1}$

Algorithm	Efficient	Authors
Forwards Additive	No	Lucas, Kanade
Forwards compositional	No	Shum, Szeliski
Inverse Additive	Yes	Hager, Belhumeur
Inverse Compositional	Yes	Baker, Matthews

Kanade-Lucas-Tomasi (KLT) tracker

https://www.youtube.com/watch?v=rwIjkECpY0M

Feature-based tracking

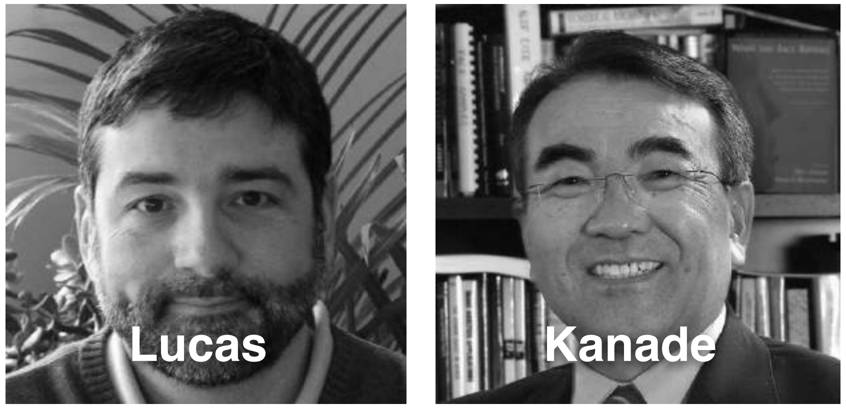
Up to now, we've been aligning entire images
but we can also track just small image regions too!
(sometimes called sparse tracking or sparse alignment)

How should we select the 'small images' (features)?

How should we track them from frame to frame?

History of the

Kanade-Lucas-Tomasi (KLT) Tracker



An Iterative Image Registration Technique with an Application to Stereo Vision.

1981

Detection and Tracking of Feature Points.

1991

The original KLT algorithm

Good Features to Track.

1994

Kanade-Lucas-Tomasi

How should we track them from frame to frame?

Lucas-Kanade

Method for aligning (tracking) an image patch

How should we select features?

Tomasi-Kanade

Method for choosing the best feature (image patch) for tracking

What are good features for tracking?

What are good features for tracking?

Intuitively, we want to avoid smooth regions and edges.
But is there a more is principled way to define good features?

What are good features for tracking?

Can be derived from the tracking algorithm

What are good features for tracking?

Can be derived from the tracking algorithm

'A feature is good if it can be tracked well'

Recall the Lucas-Kanade image alignment method:

error function (SSD)

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

incremental update

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

Recall the Lucas-Kanade image alignment method:

error function (SSD)
$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

incremental update
$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

linearize
$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Recall the Lucas-Kanade image alignment method:

error function (SSD)

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

incremental update

$$\sum_{\boldsymbol{x}} [I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x})]^2$$

linearize

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x})\right]^2$$

Gradient update

$$\Delta\boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^\top [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$$

$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^\top \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]$$

Recall the Lucas-Kanade image alignment method:

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$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

incremental update

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}+\Delta\boldsymbol{p})) - T(\boldsymbol{x}) \right]^2$$

linearize

$$\sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta\boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

Gradient update

$$\Delta\boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top \left[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) \right]$$

$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

Update

$$\boldsymbol{p} \leftarrow \boldsymbol{p} + \Delta\boldsymbol{p}$$

Stability of gradient decent iterations depends on …

$$\Delta \boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} \left[T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p})) \right]$$

Stability of gradient decent iterations depends on …

$$\Delta \boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$$

Inverting the Hessian

$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

When does the inversion fail?

Stability of gradient decent iterations depends on …

$$\Delta \boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x}; \boldsymbol{p}))]$$

Inverting the Hessian

$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^{\top} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

When does the inversion fail?

H is singular. But what does that mean?

Above the noise level

$$\lambda_1 \gg 0$$

$$\lambda_2 \gg 0$$

both Eigenvalues are large

Well-conditioned

both Eigenvalues have similar magnitude

Concrete example: Consider translation model

$$\mathbf{W}(\boldsymbol{x};\boldsymbol{p}) = \begin{bmatrix} x + p_1 \\ y + p_2 \end{bmatrix} \qquad \frac{\mathbf{W}}{\partial \boldsymbol{p}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hessian

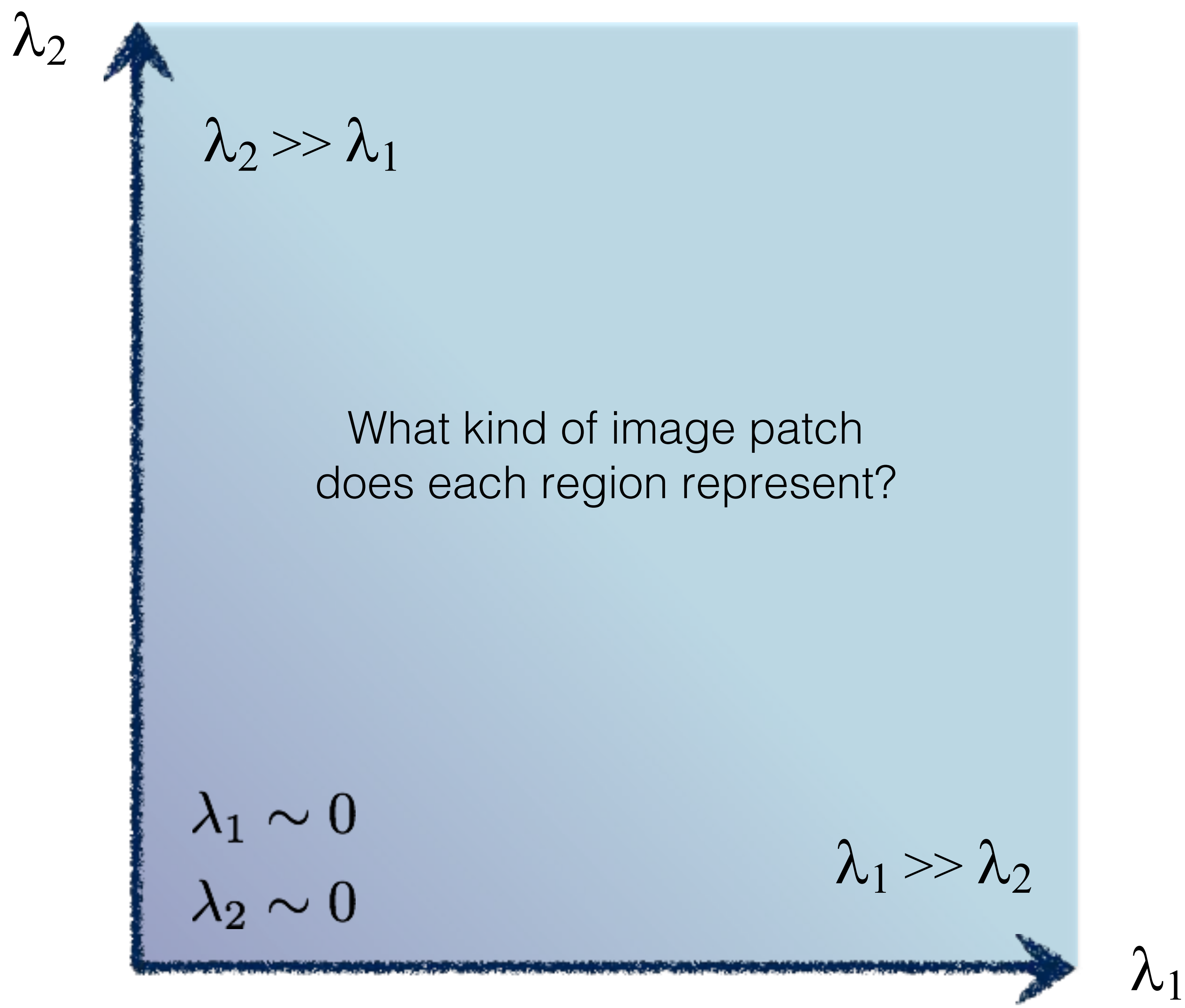
$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]^{\top} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}\right]$$

$$= \sum_{\boldsymbol{x}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_x \\ I_y \end{bmatrix} \begin{bmatrix} I_x & I_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

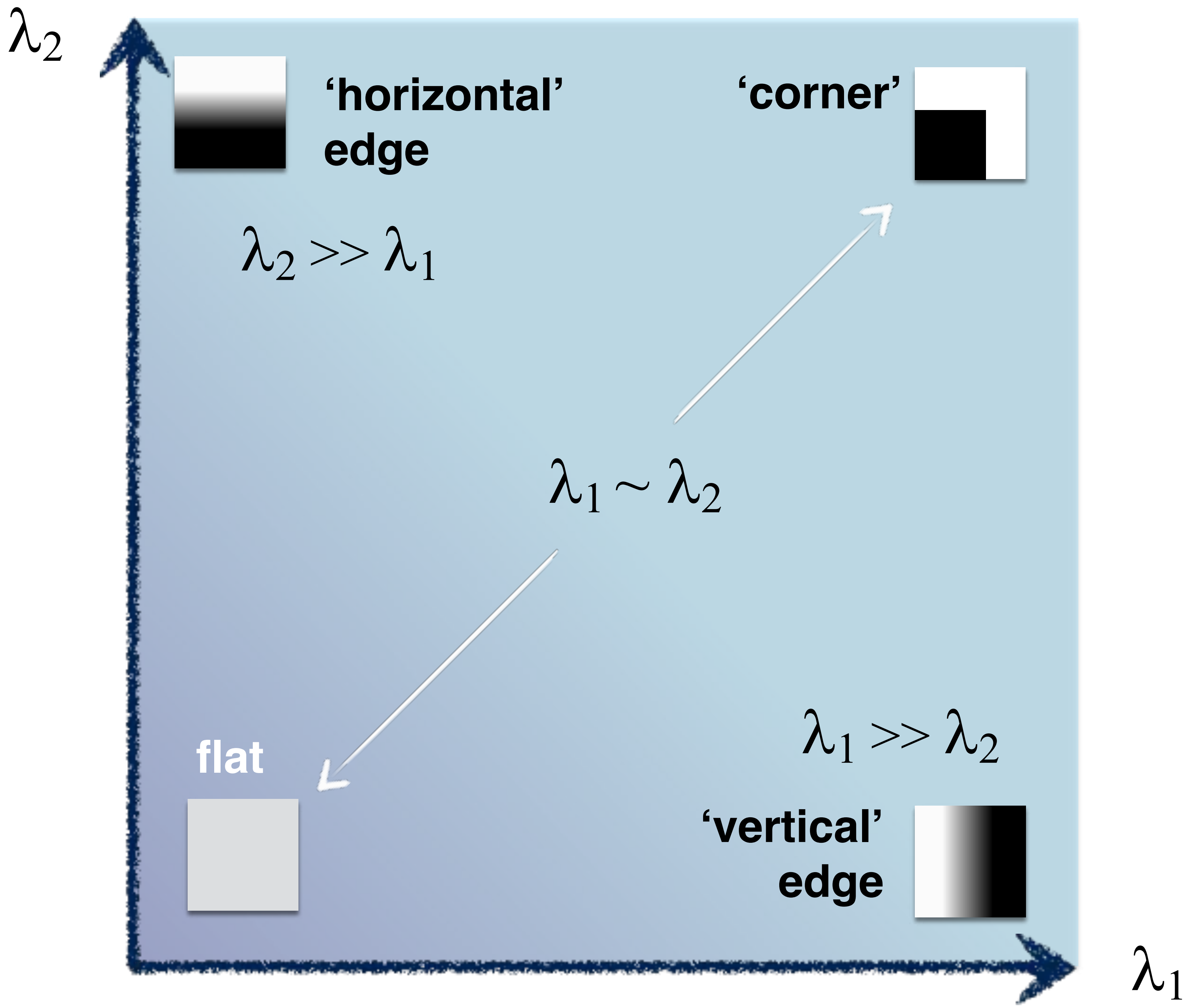
$$= \begin{bmatrix} \sum_{\boldsymbol{x}} I_x I_x & \sum_{\boldsymbol{x}} I_y I_x \\ \sum_{\boldsymbol{x}} I_x I_y & \sum_{\boldsymbol{x}} I_y I_y \end{bmatrix}$$ ←*when is this singular?*

How are the eigenvalues related to image content?

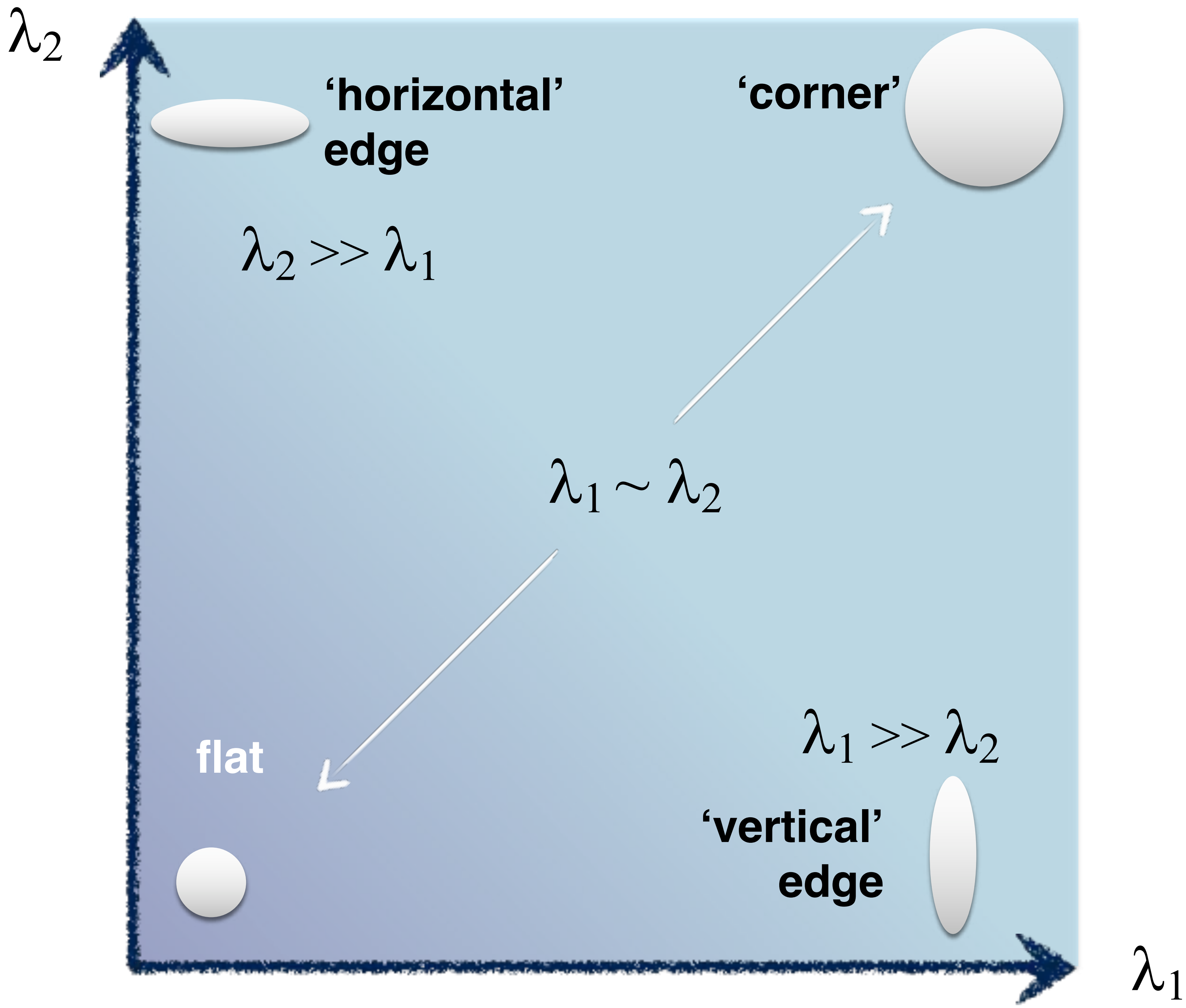
interpreting eigenvalues



interpreting eigenvalues



interpreting eigenvalues



What are good features for tracking?

What are good features for tracking?

$$\min(\lambda_1, \lambda_2) > \lambda$$

'big Eigenvalues means good for tracking'

KLT algorithm

1. Find corners satisfying $\min(\lambda_1, \lambda_2) > \lambda$

2. For each corner compute displacement to next frame using the Lucas-Kanade method

3. Store displacement of each corner, update corner position

4. (optional) Add more corner points every M frames using 1

5. Repeat 2 to 3 (4)

6. Returns long trajectories for each corner point

Mean-shift algorithm


Nationale-Nederlanden
jackpot €5 milj.
ING

Mean Shift Algorithm

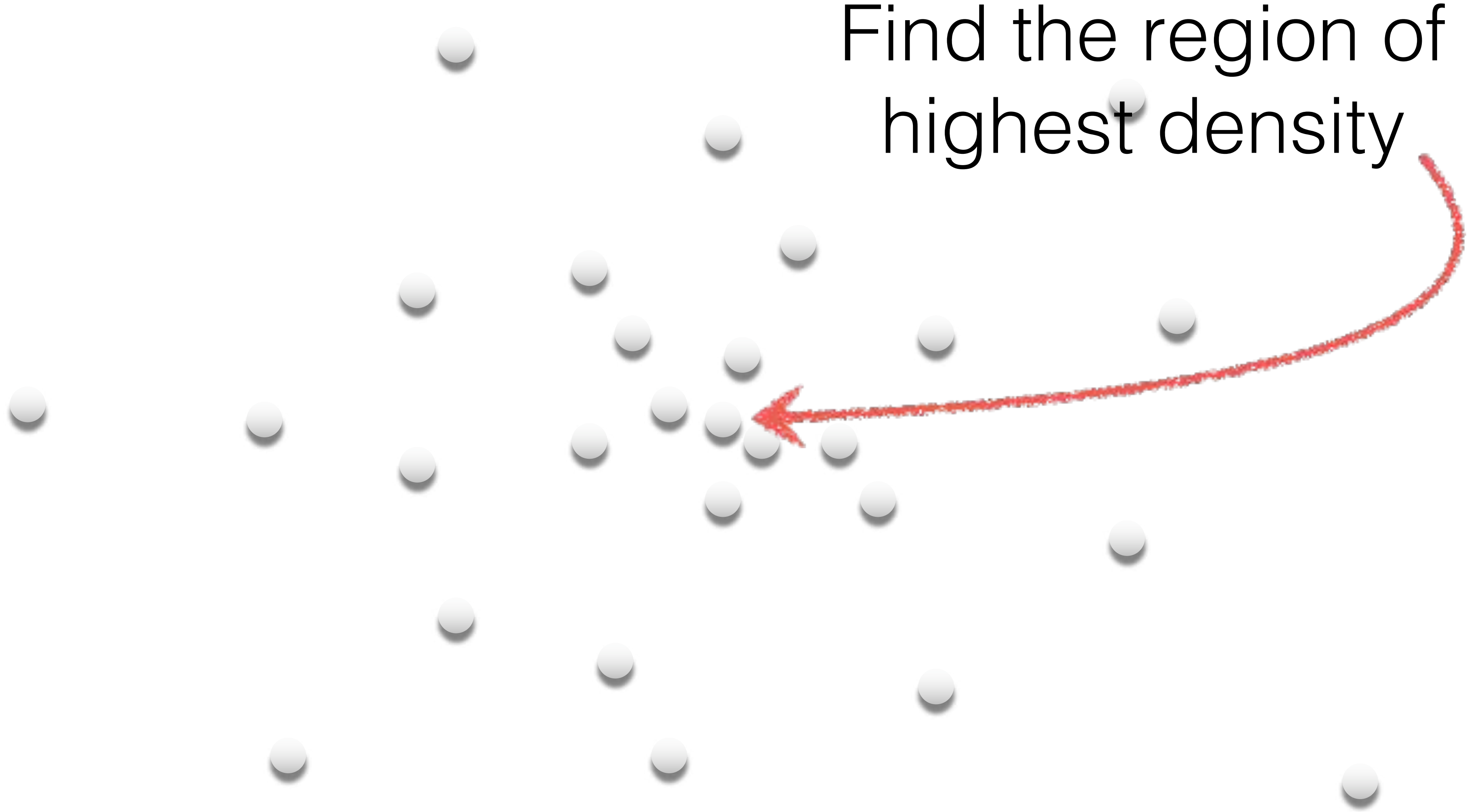
A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)



Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

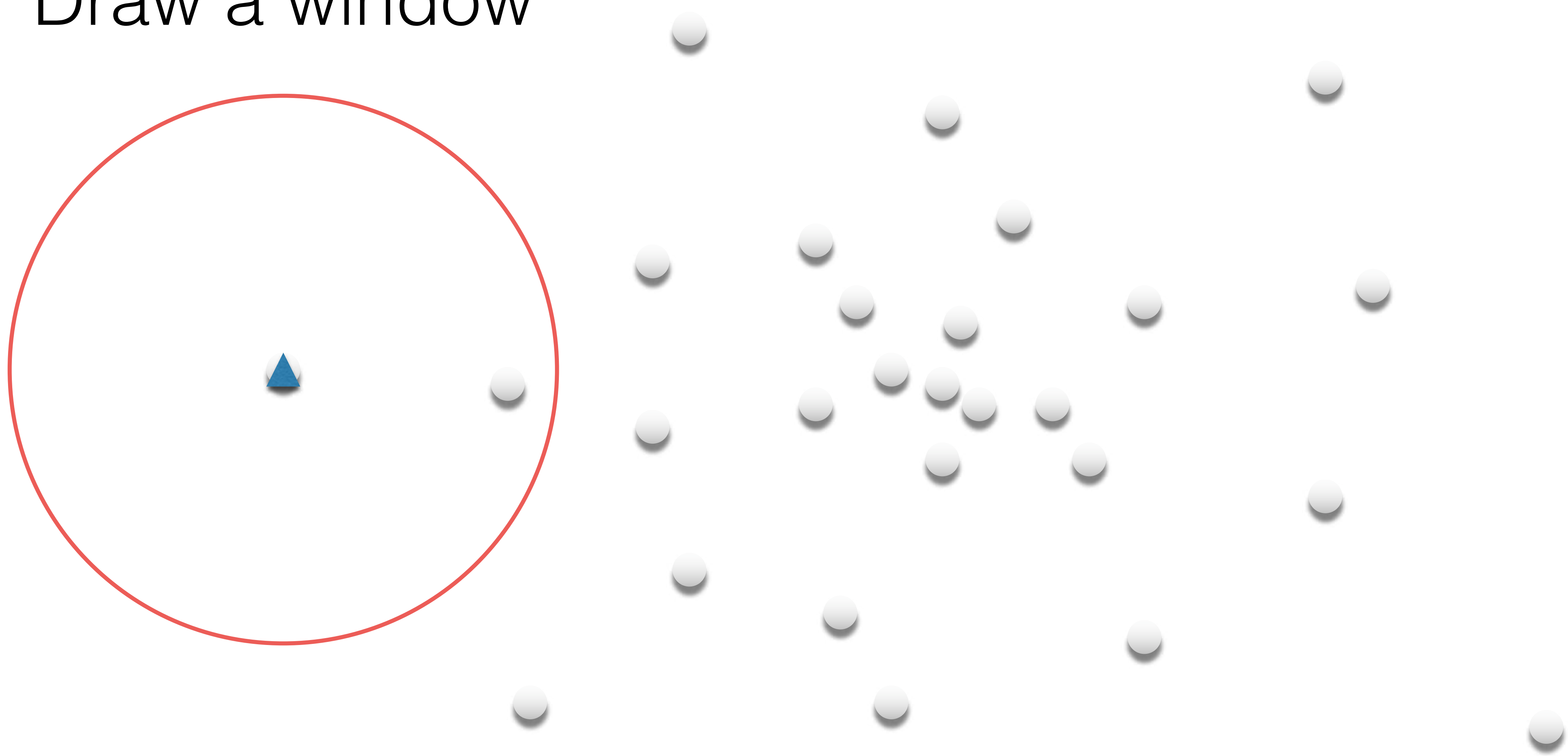
Pick a point

Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

Draw a window

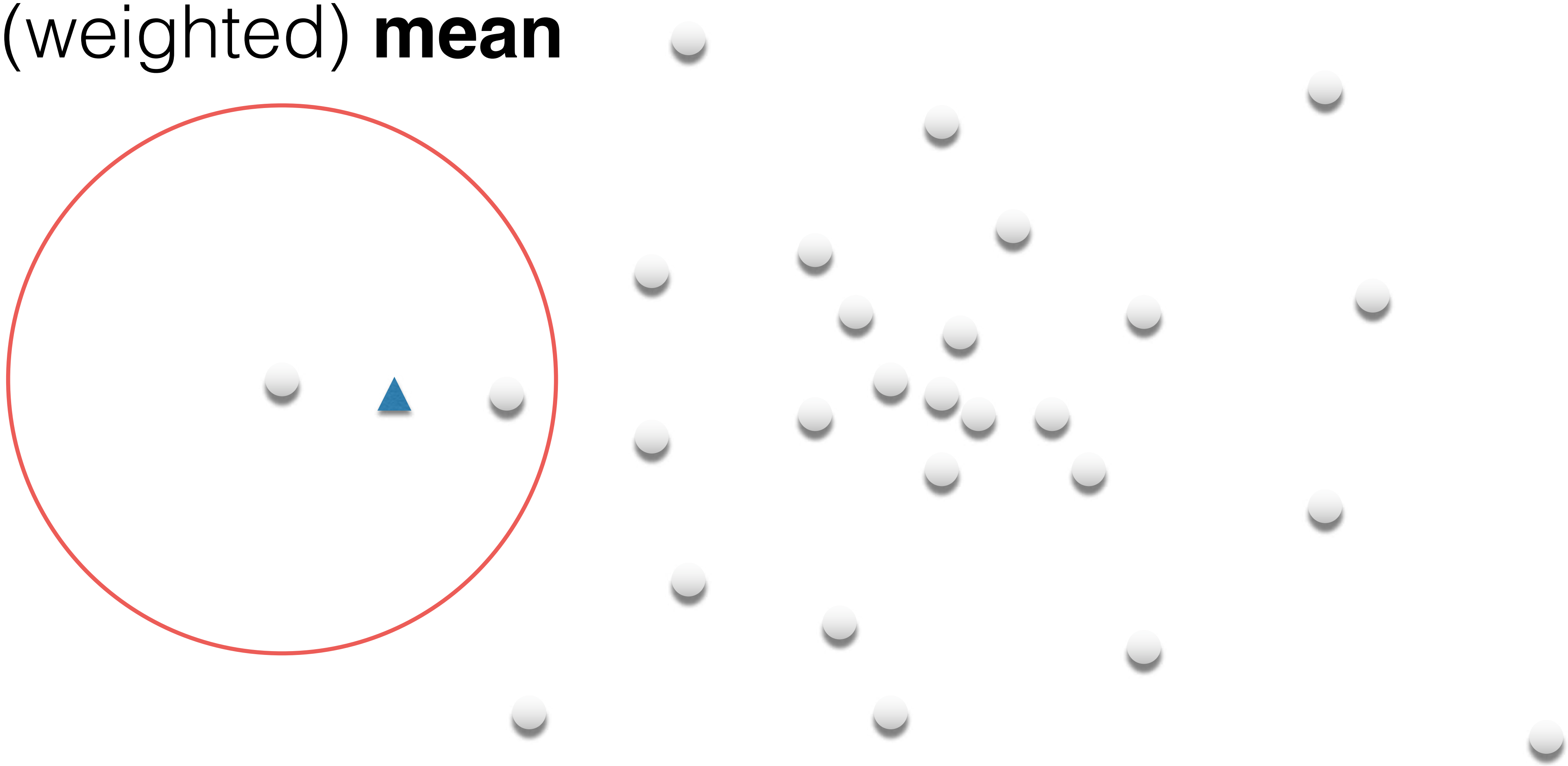


Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

Compute the
(weighted) **mean**

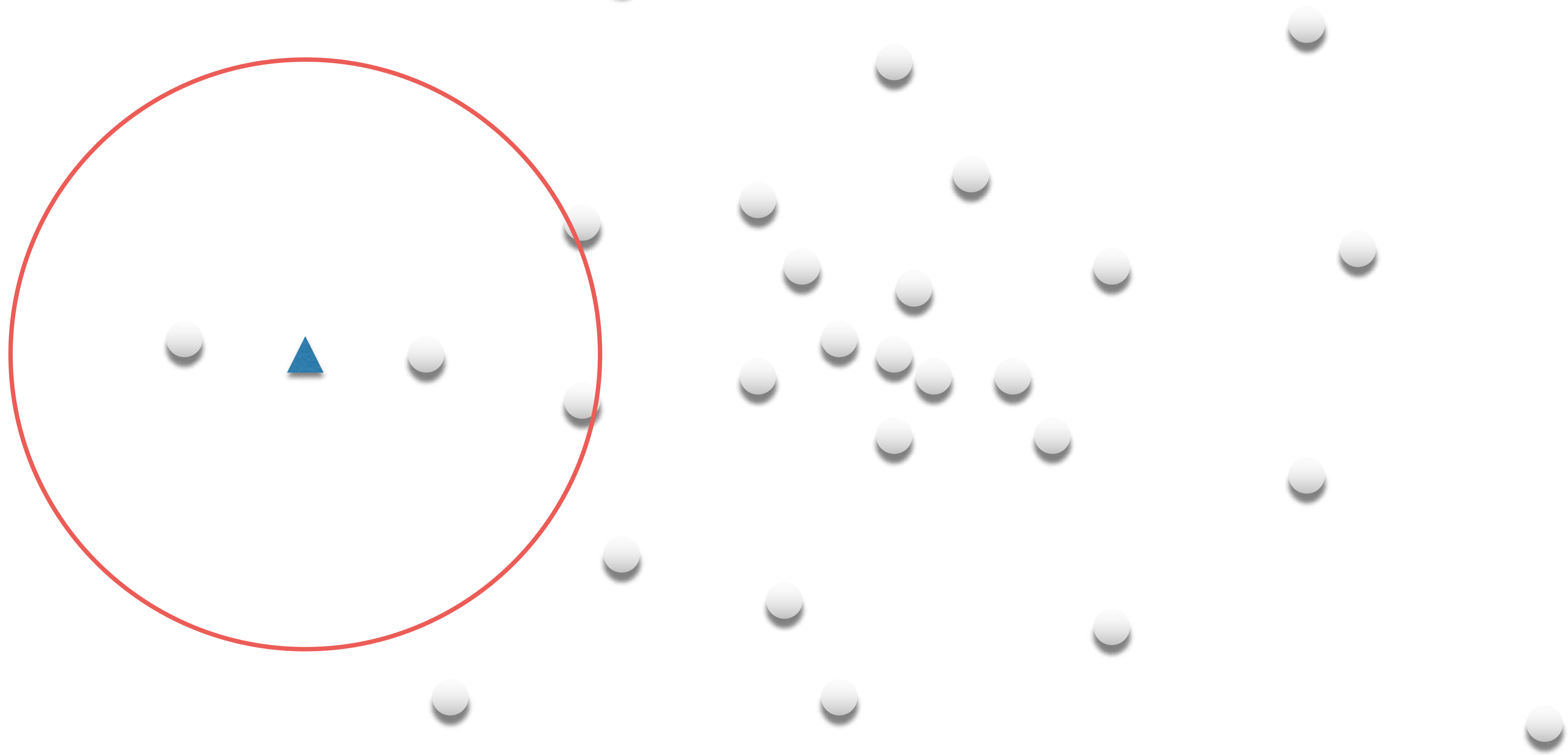


Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

Shift the window

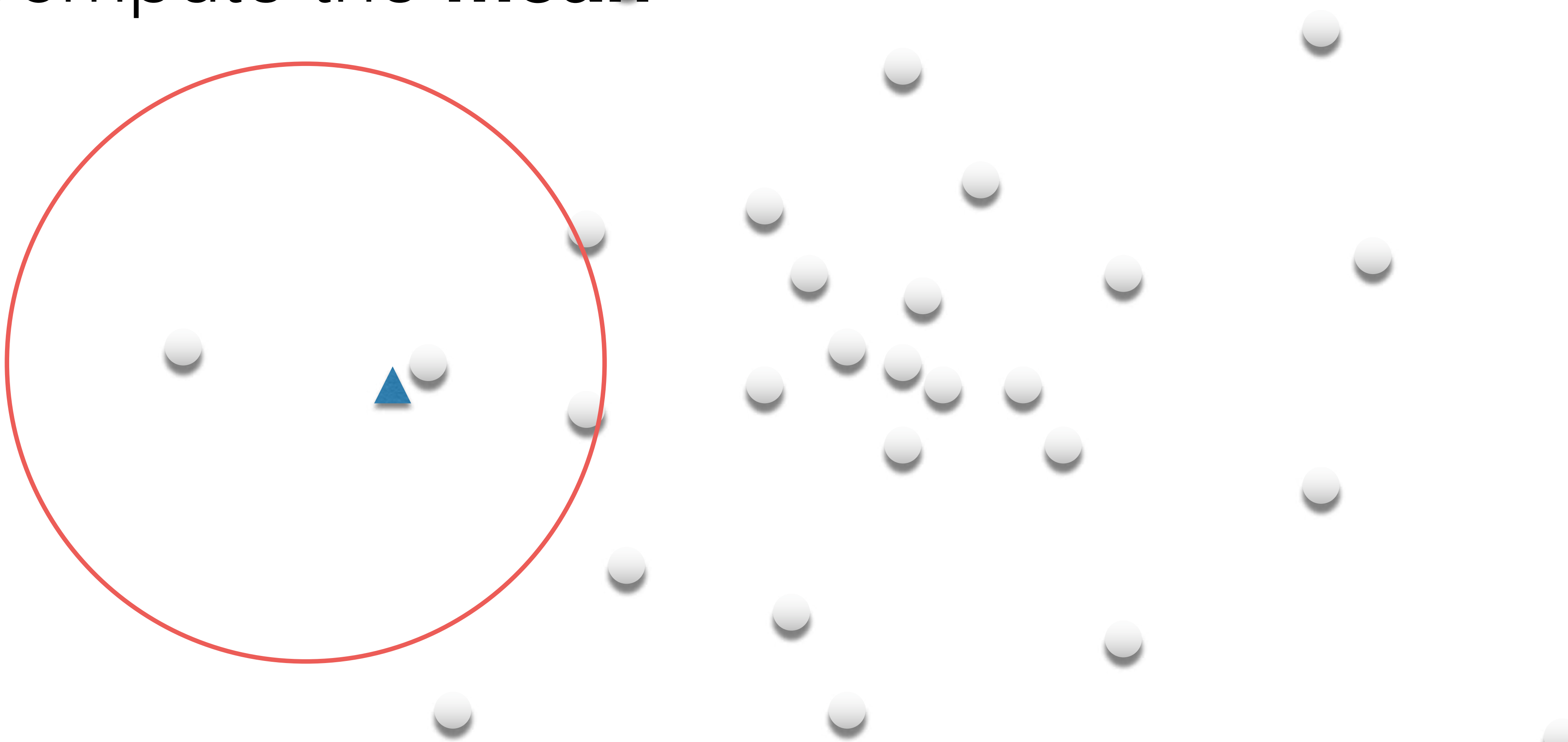


Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

Compute the **mean**



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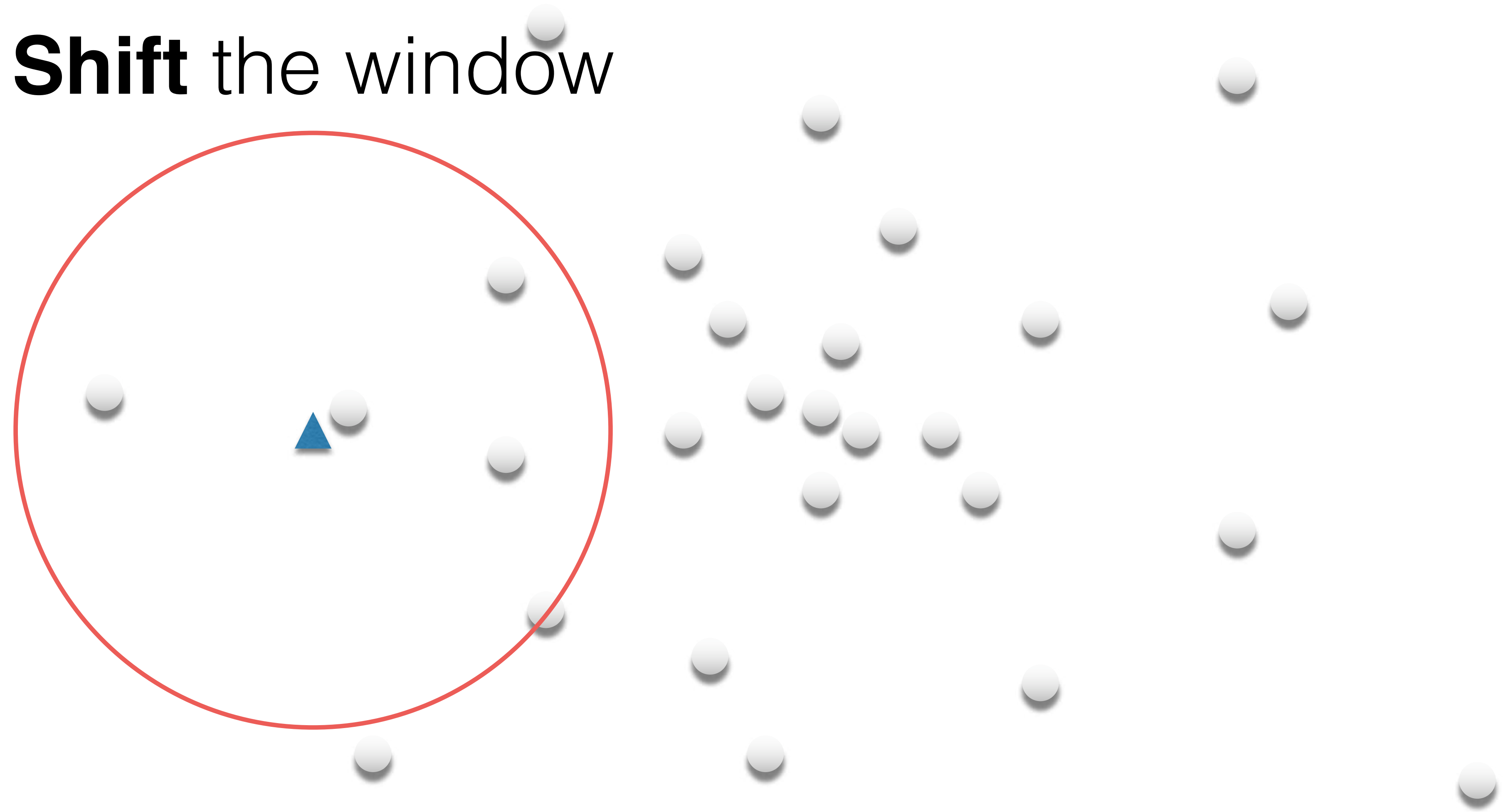

Mean Shift Algorithm

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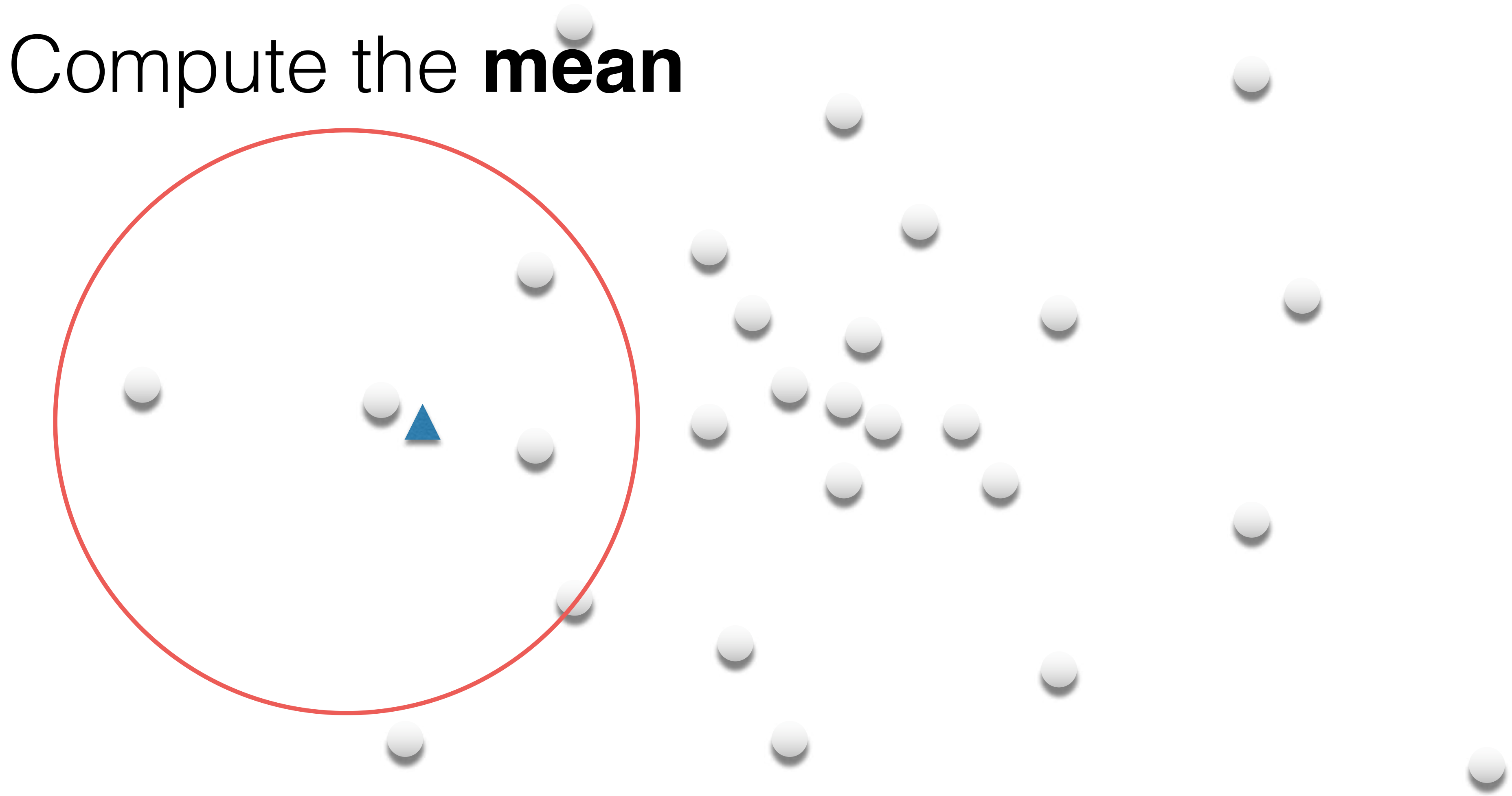

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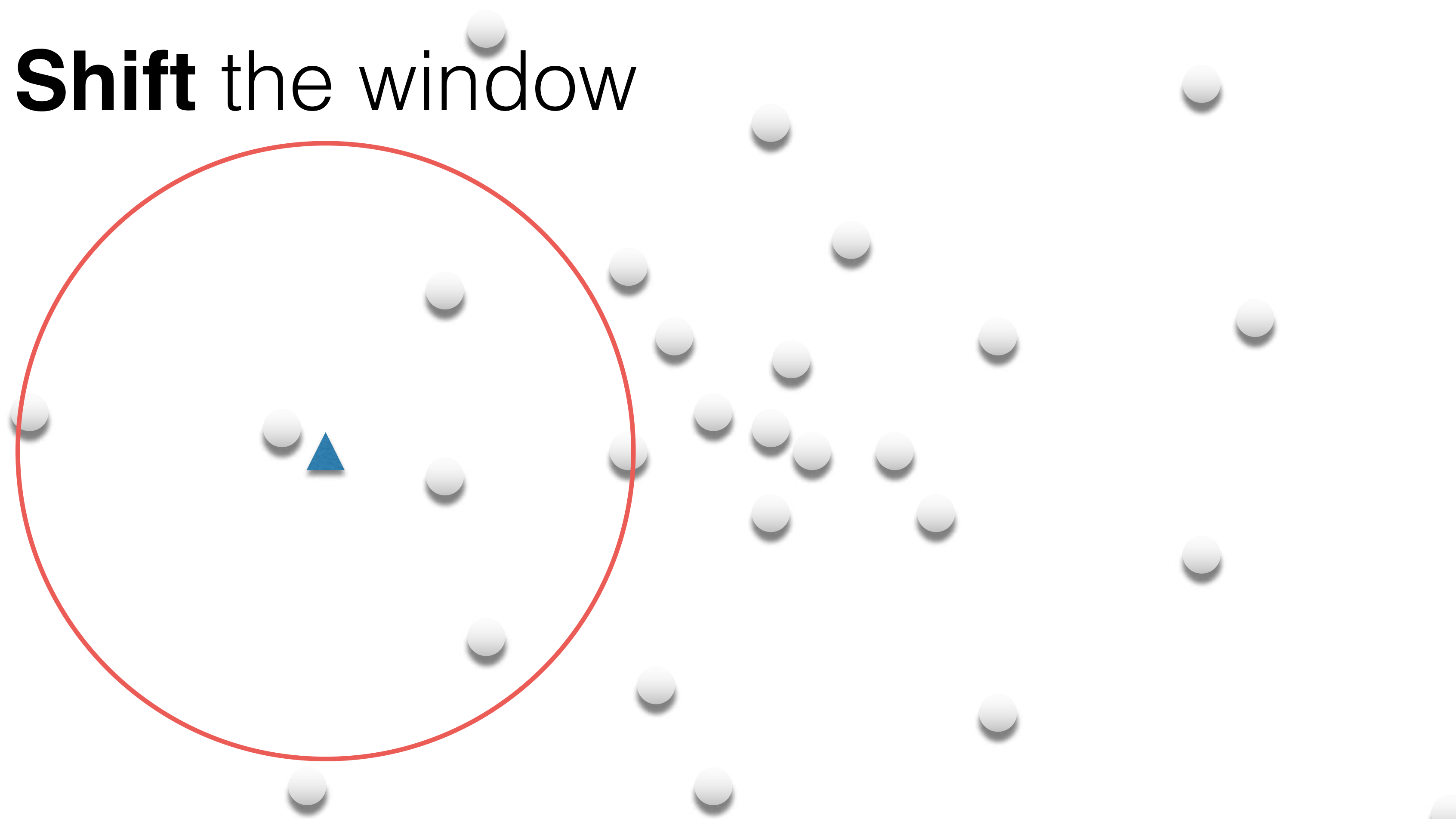

Mean Shift Algorithm

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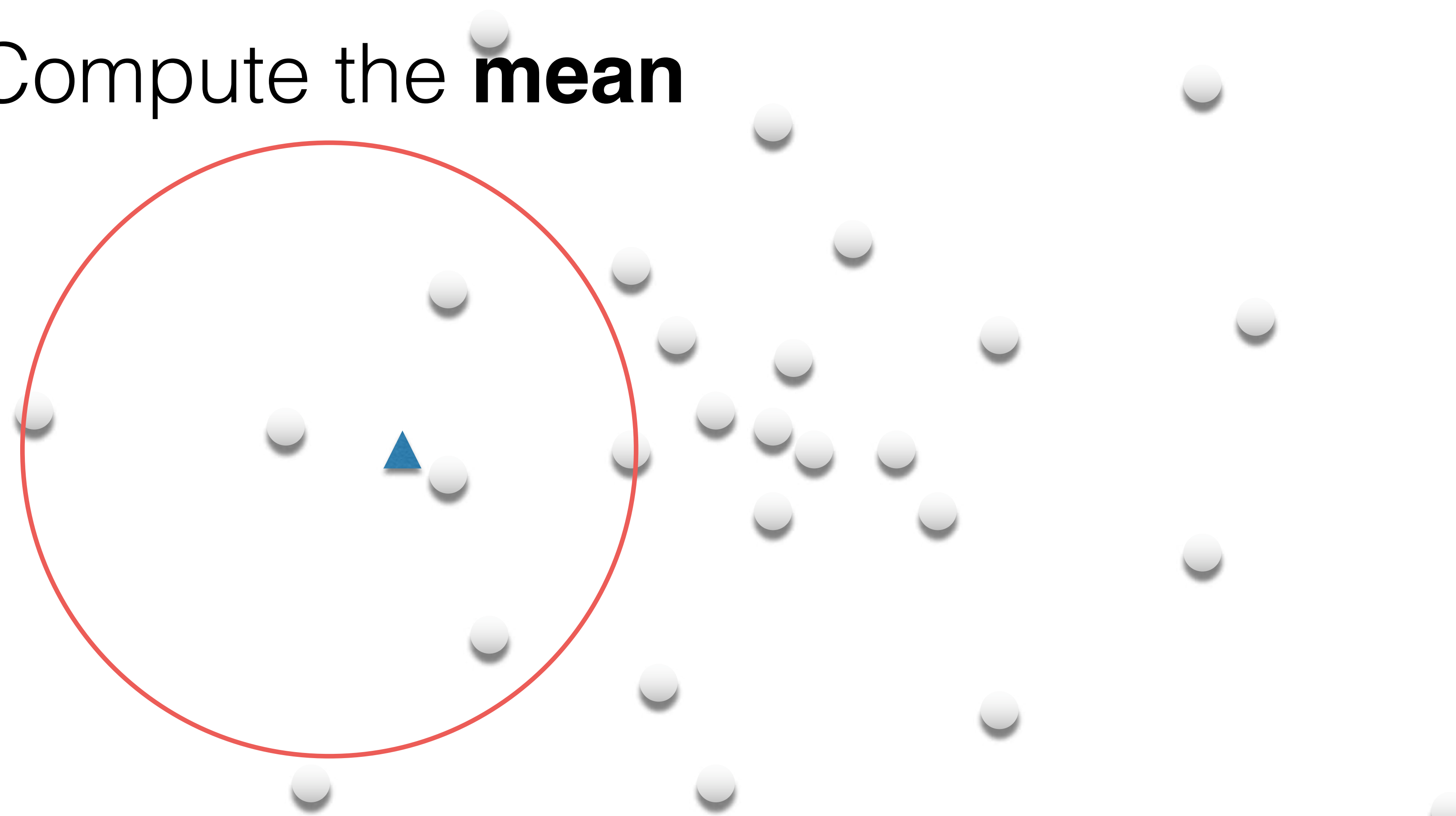

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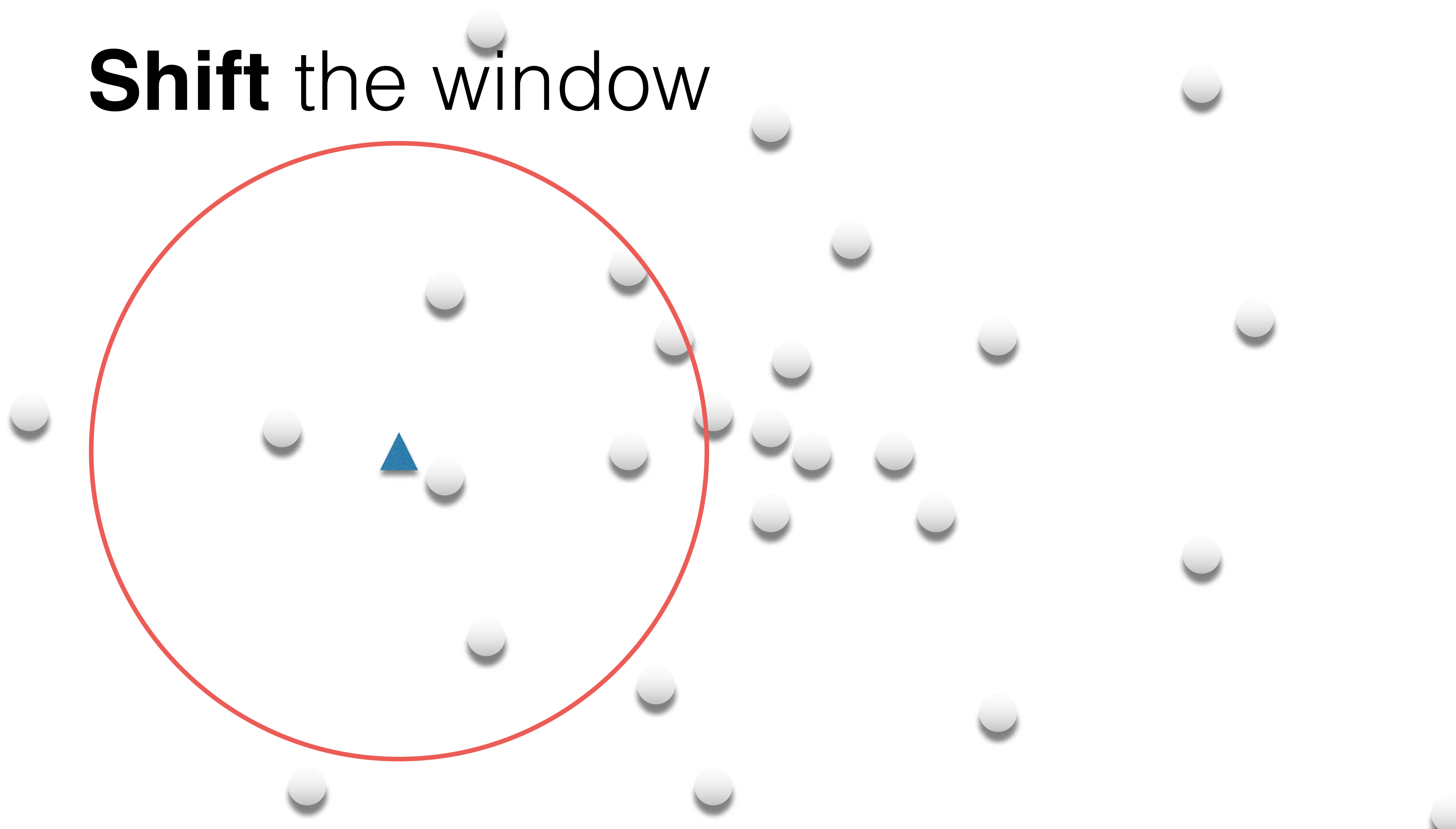
Fukunaga & Hostetler (1975)



Mean Shift Algorithm

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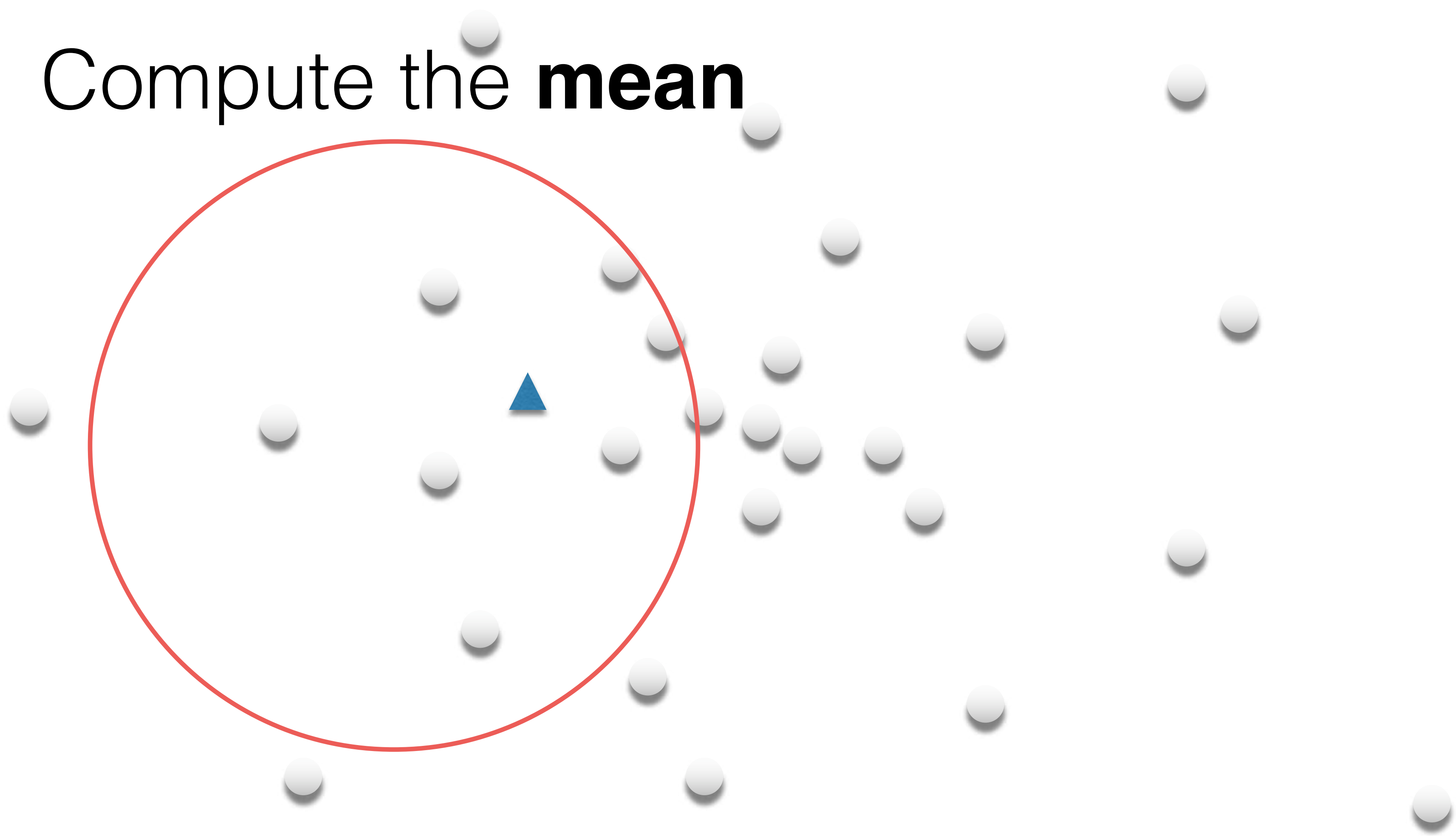
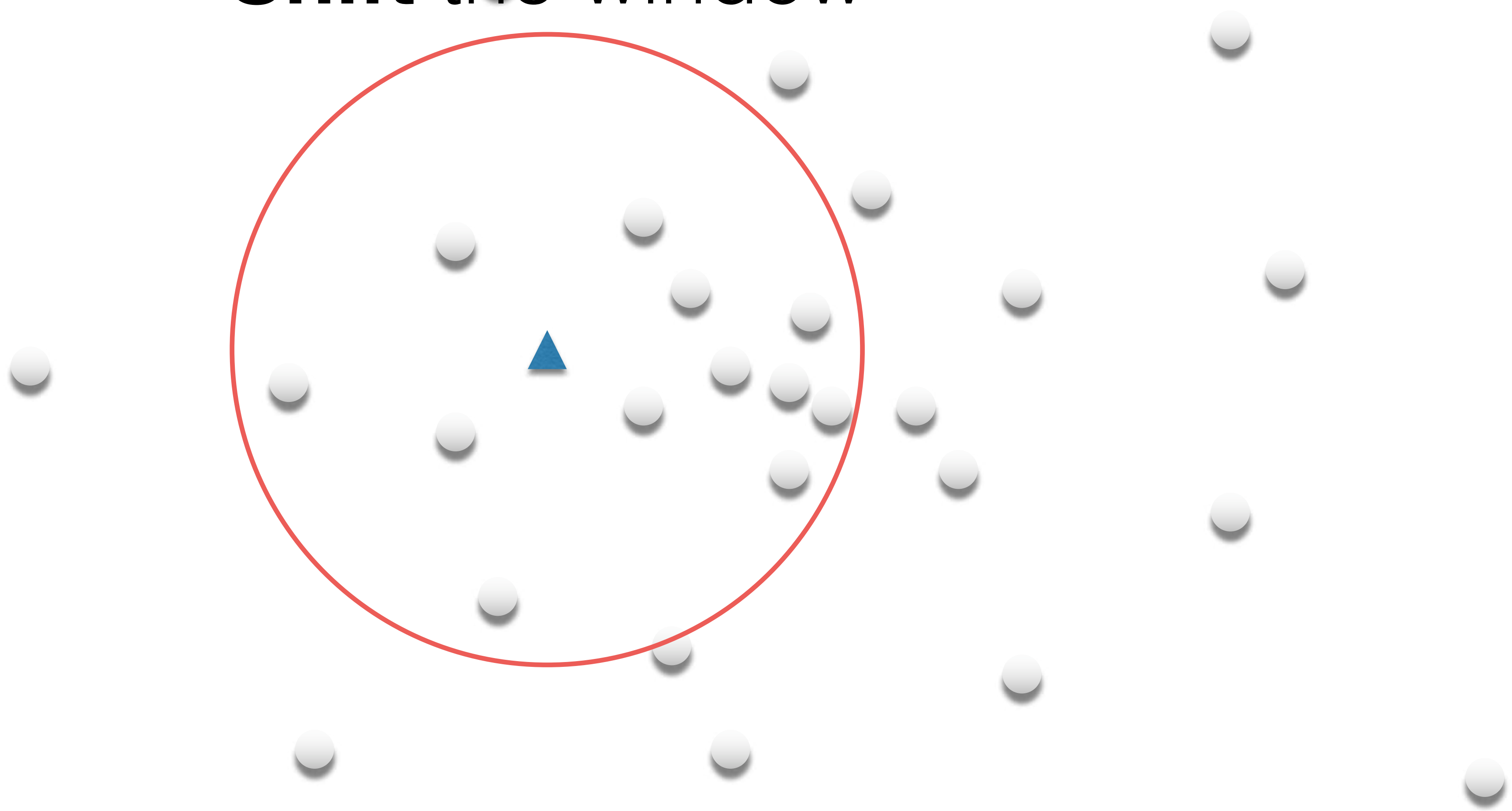


Mean Shift Algorithm

A ‘mode seeking’ algorithm

Fukunaga & Hostetler (1975)

Shift the window



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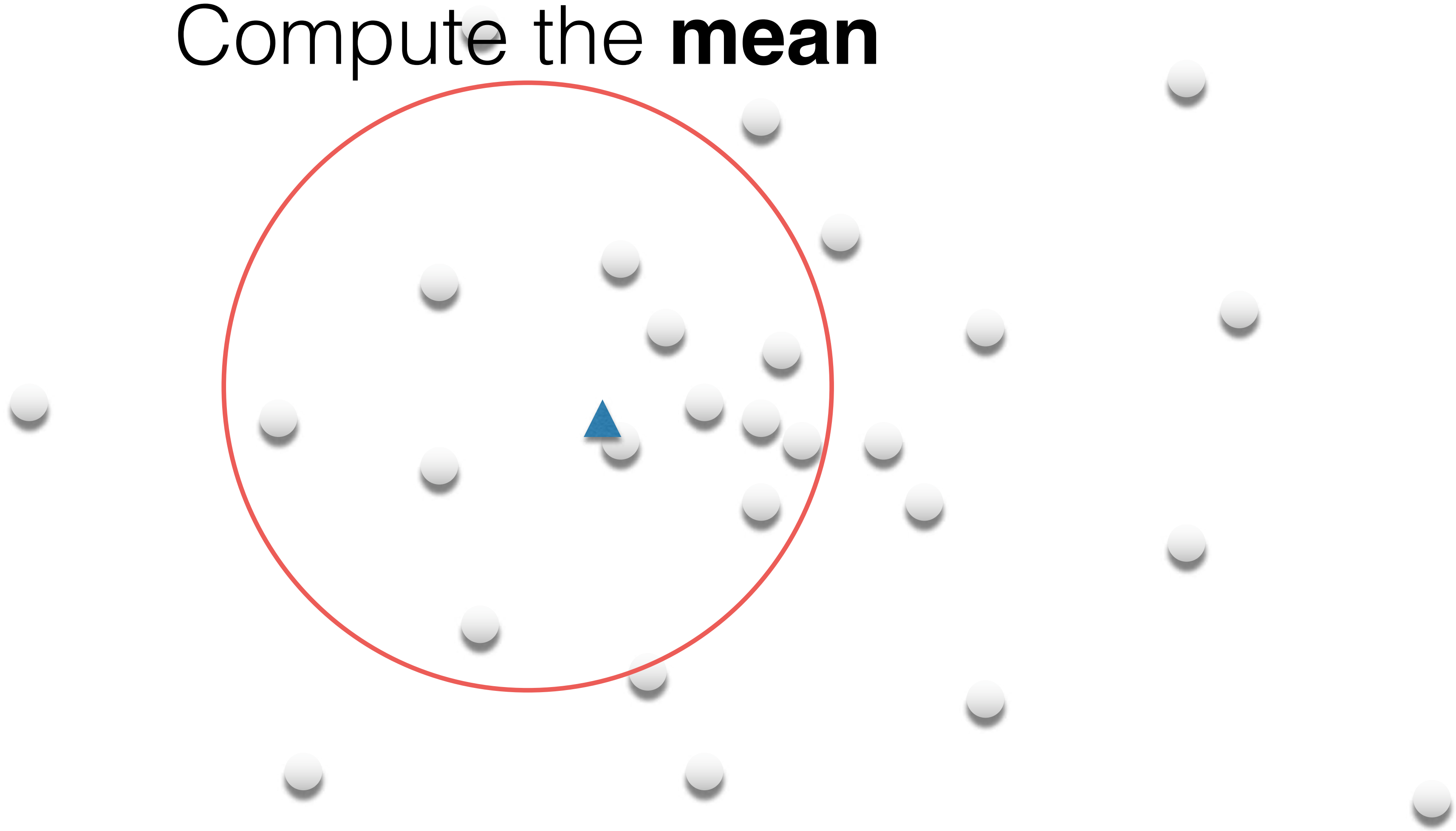

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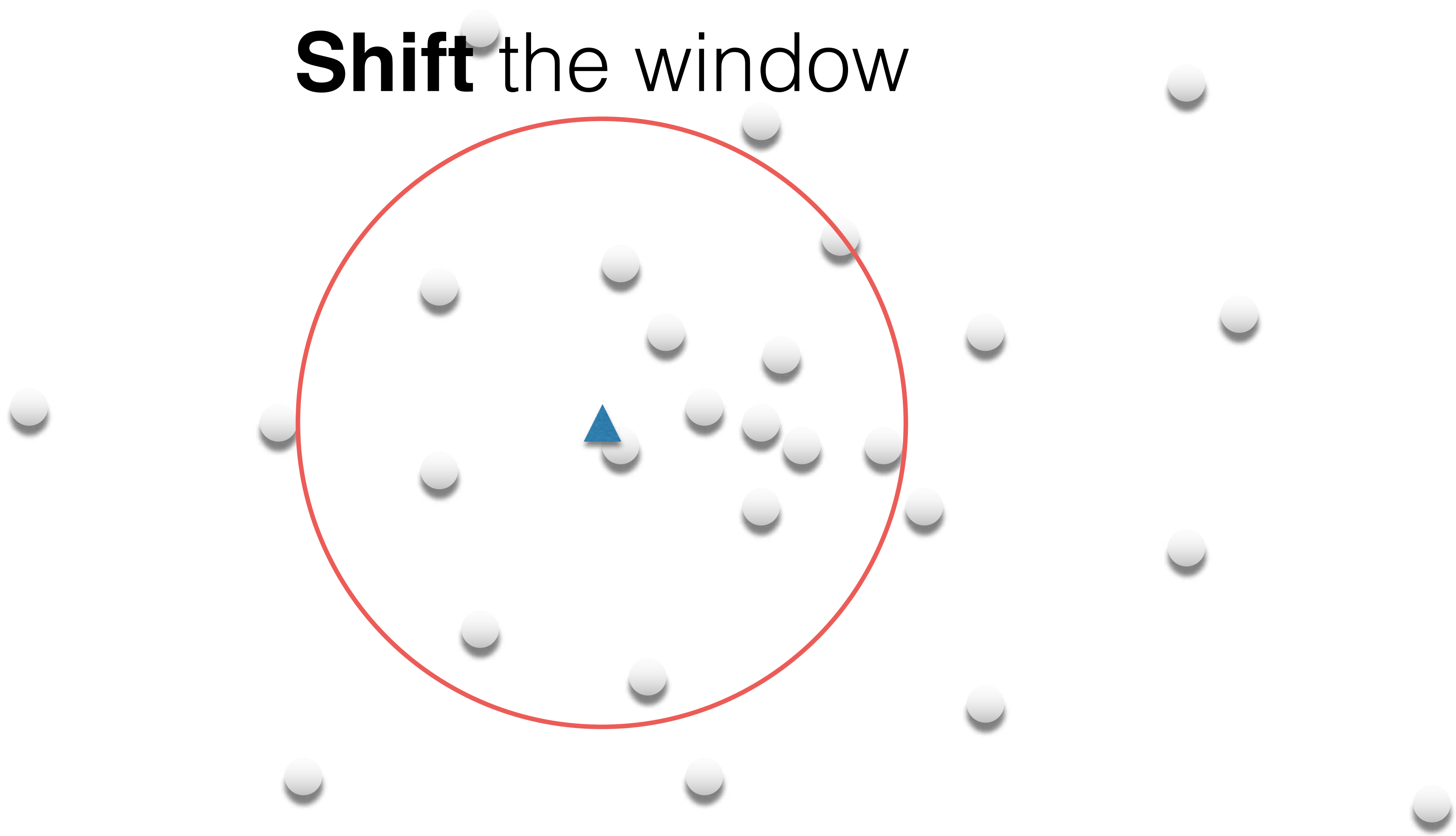

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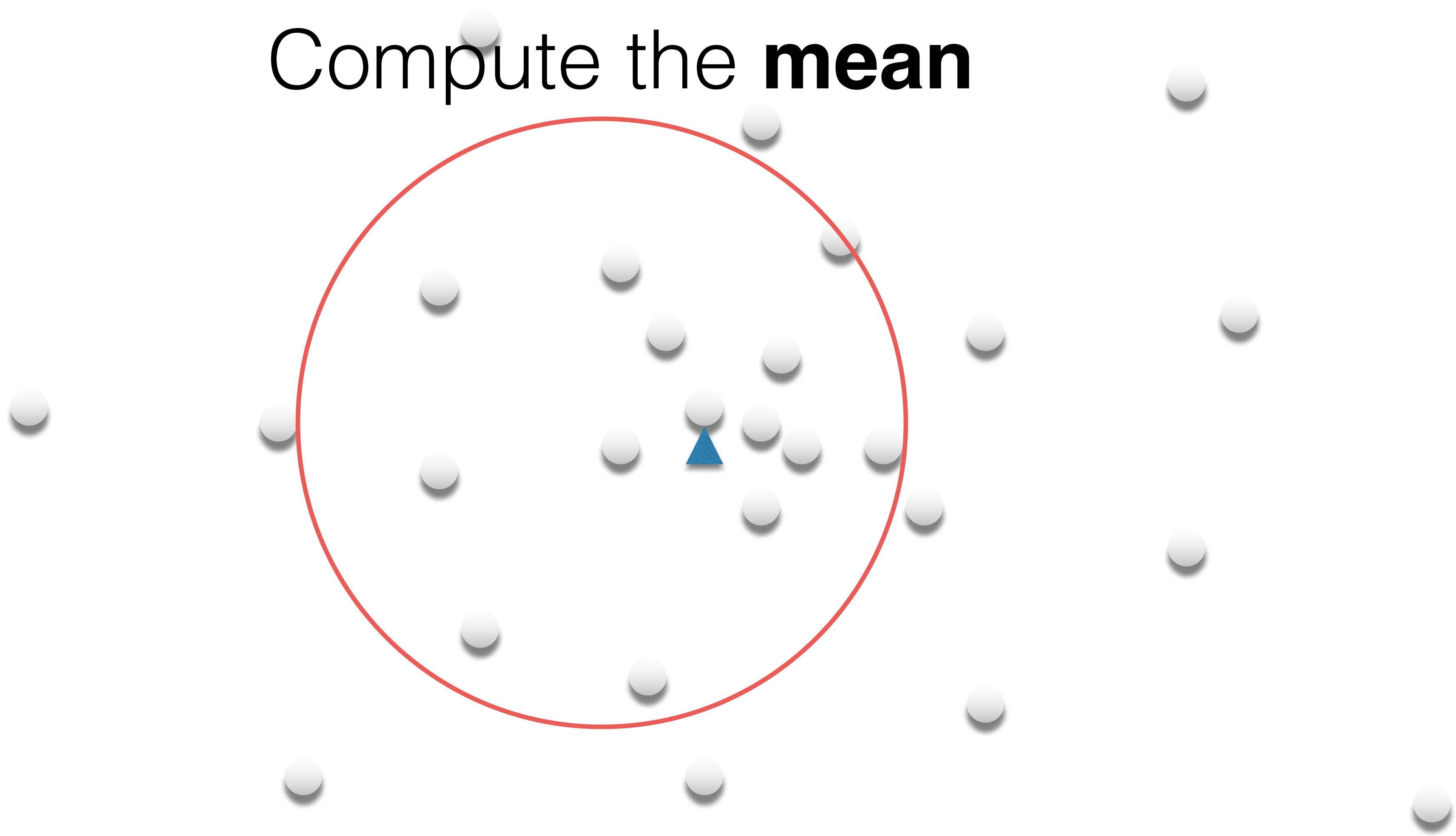

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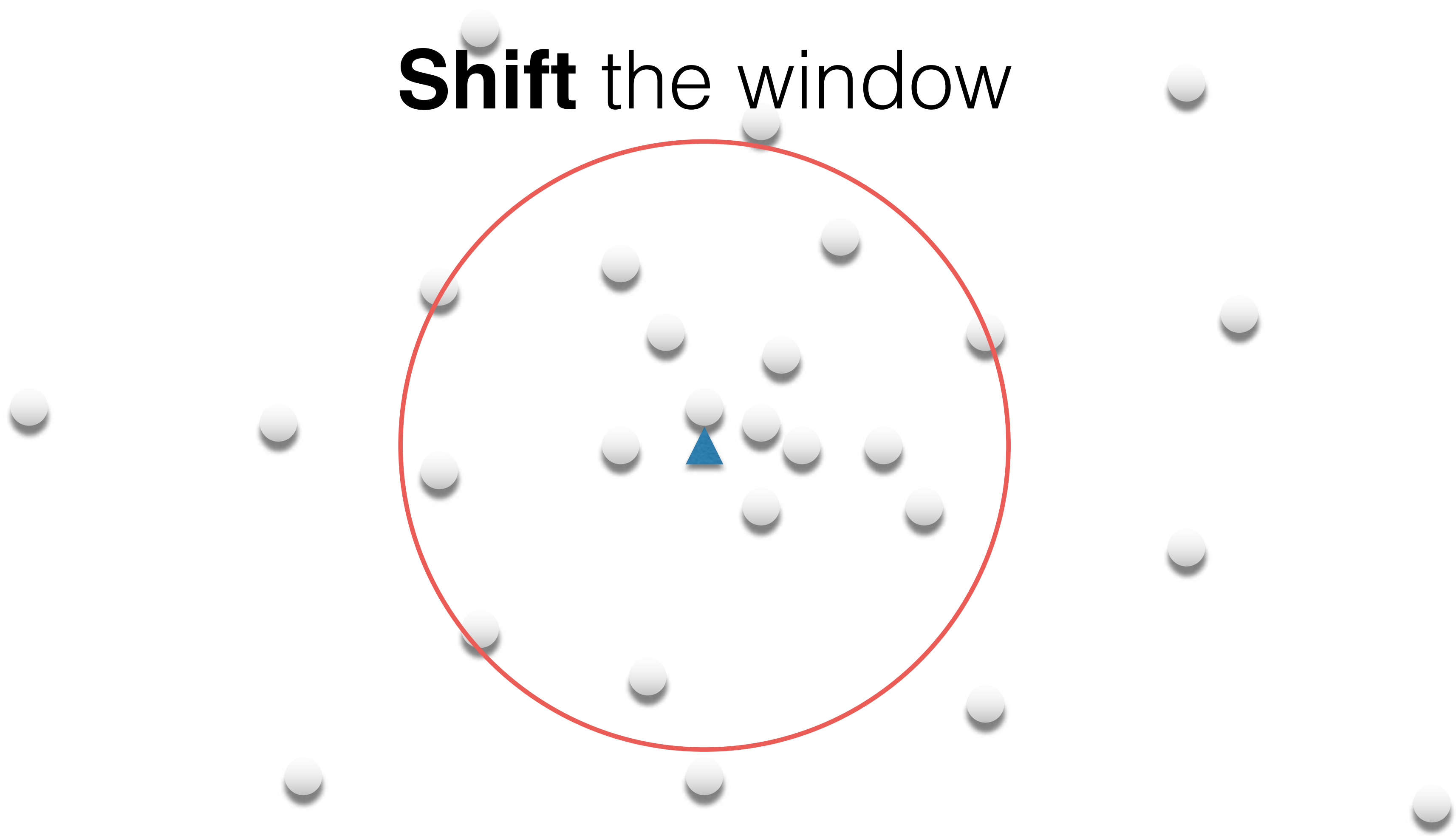

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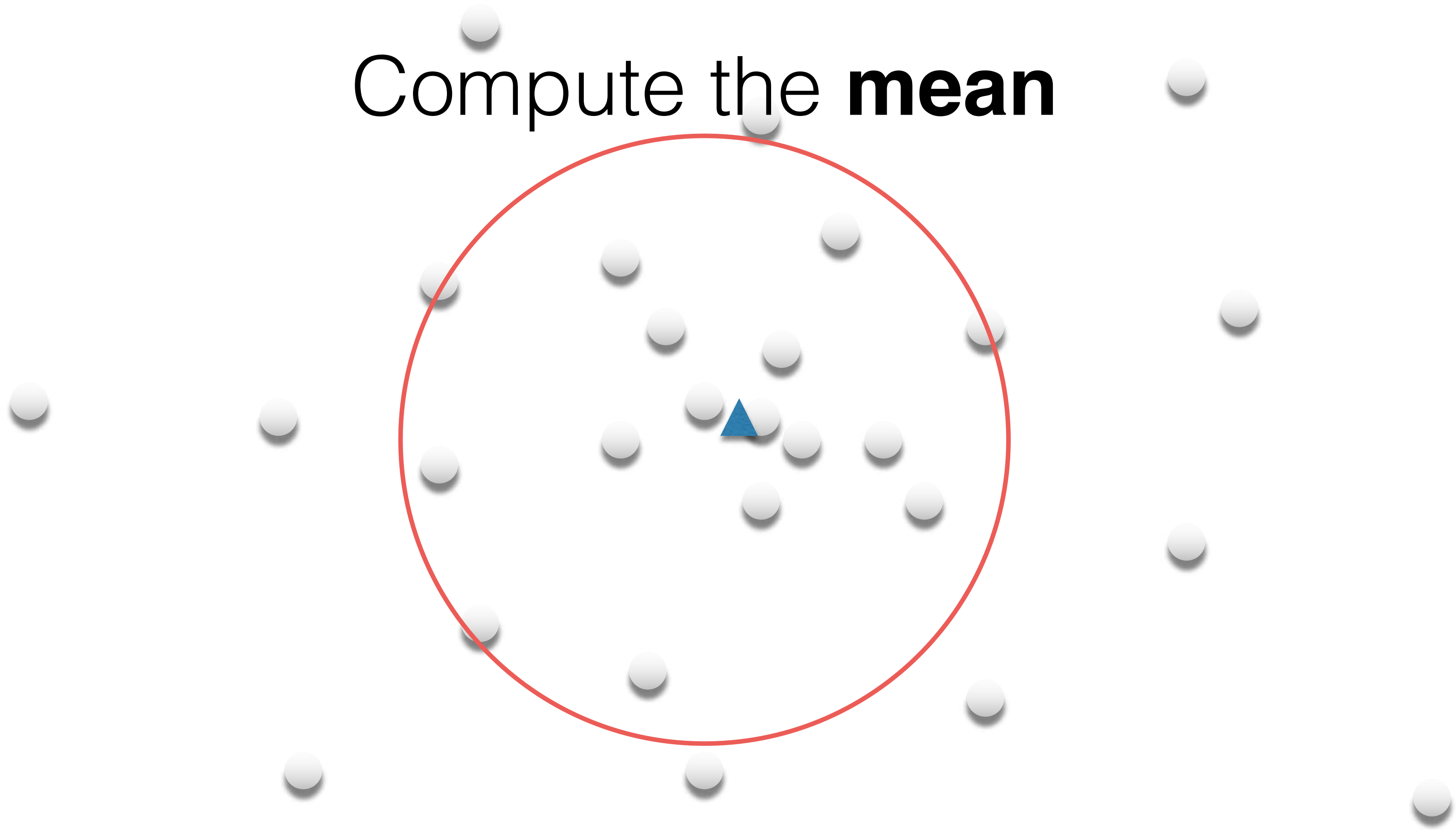

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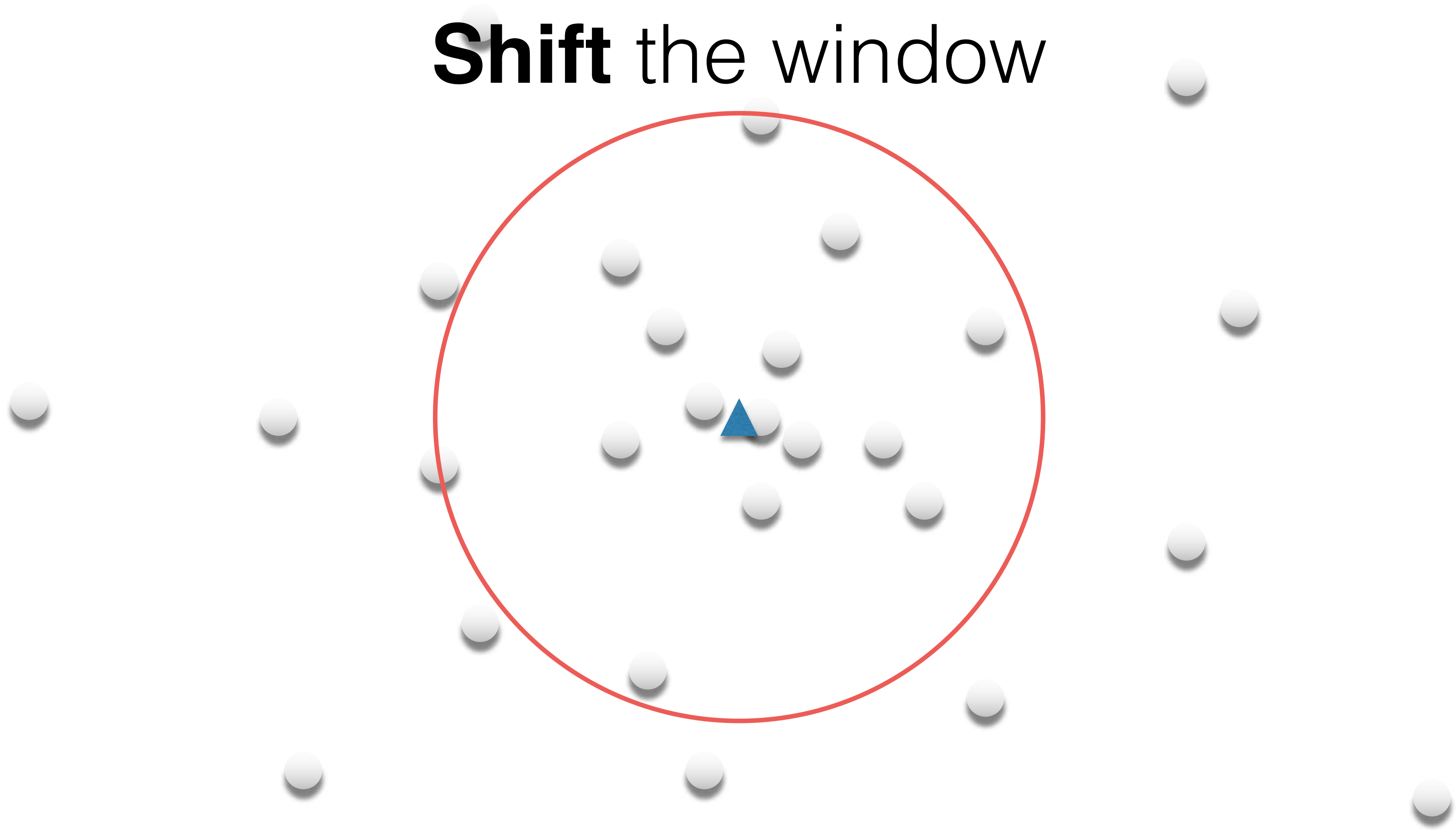


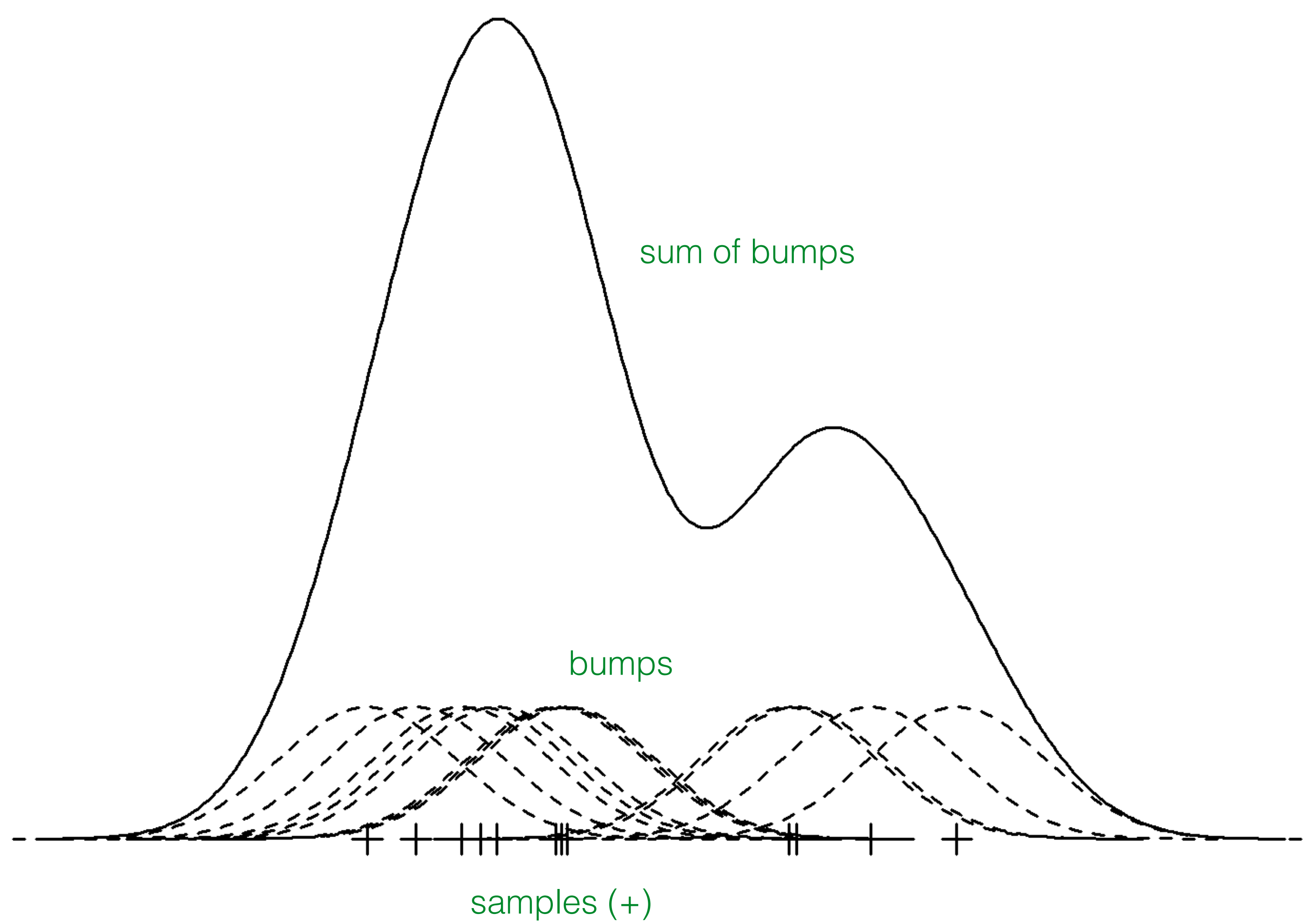
To understand the theory behind this we need to understand…

Kernel density estimation

To understand the mean shift algorithm …

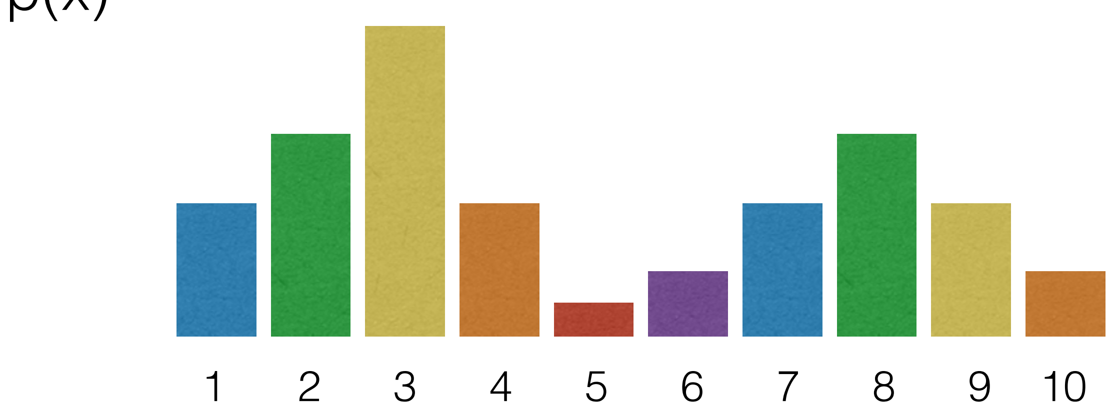
Kernel Density Estimation

A method to approximate an underlying PDF from samples

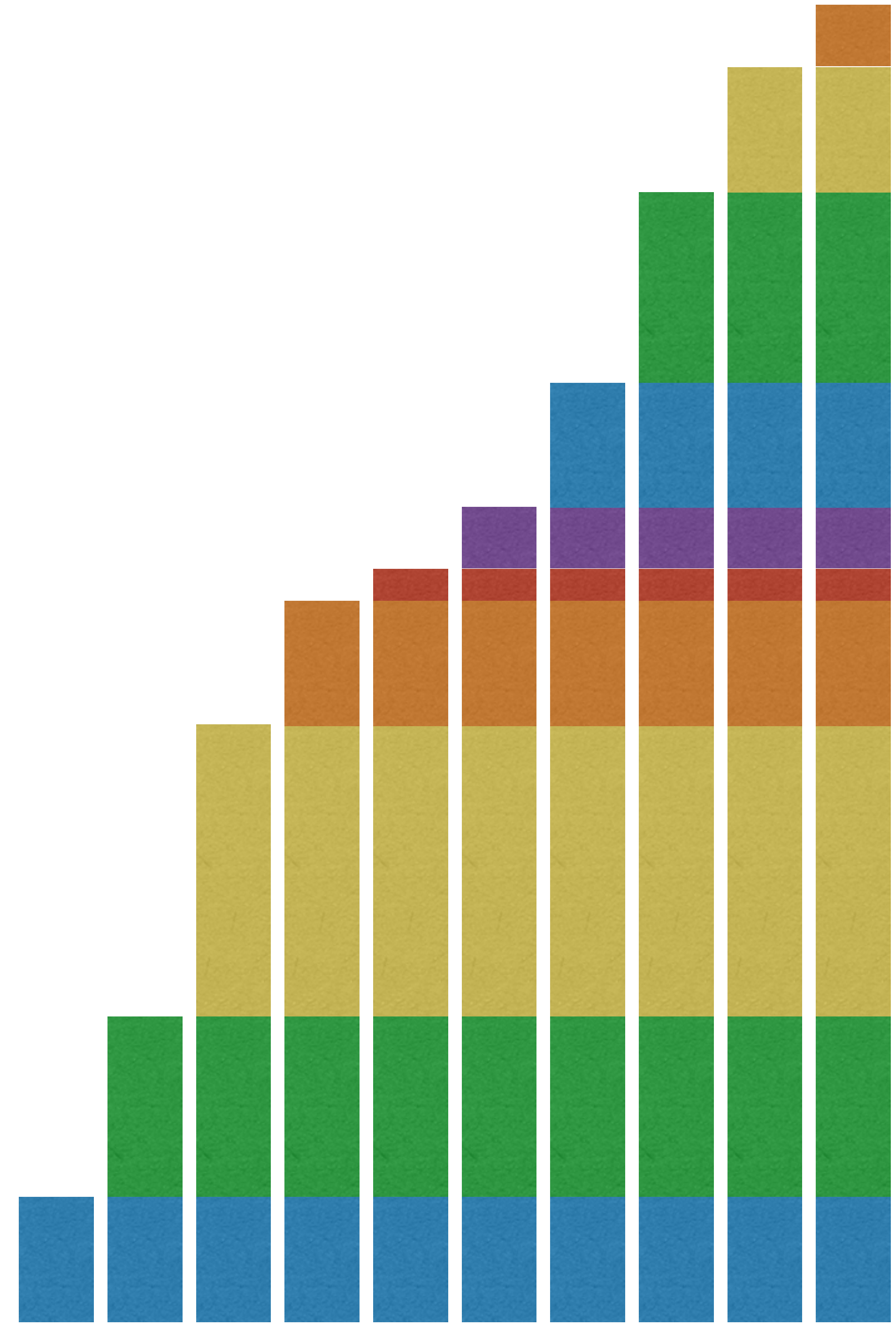


Put ‘bump’ on every sample to approximate the PDF

Say we have some hidden PDF…

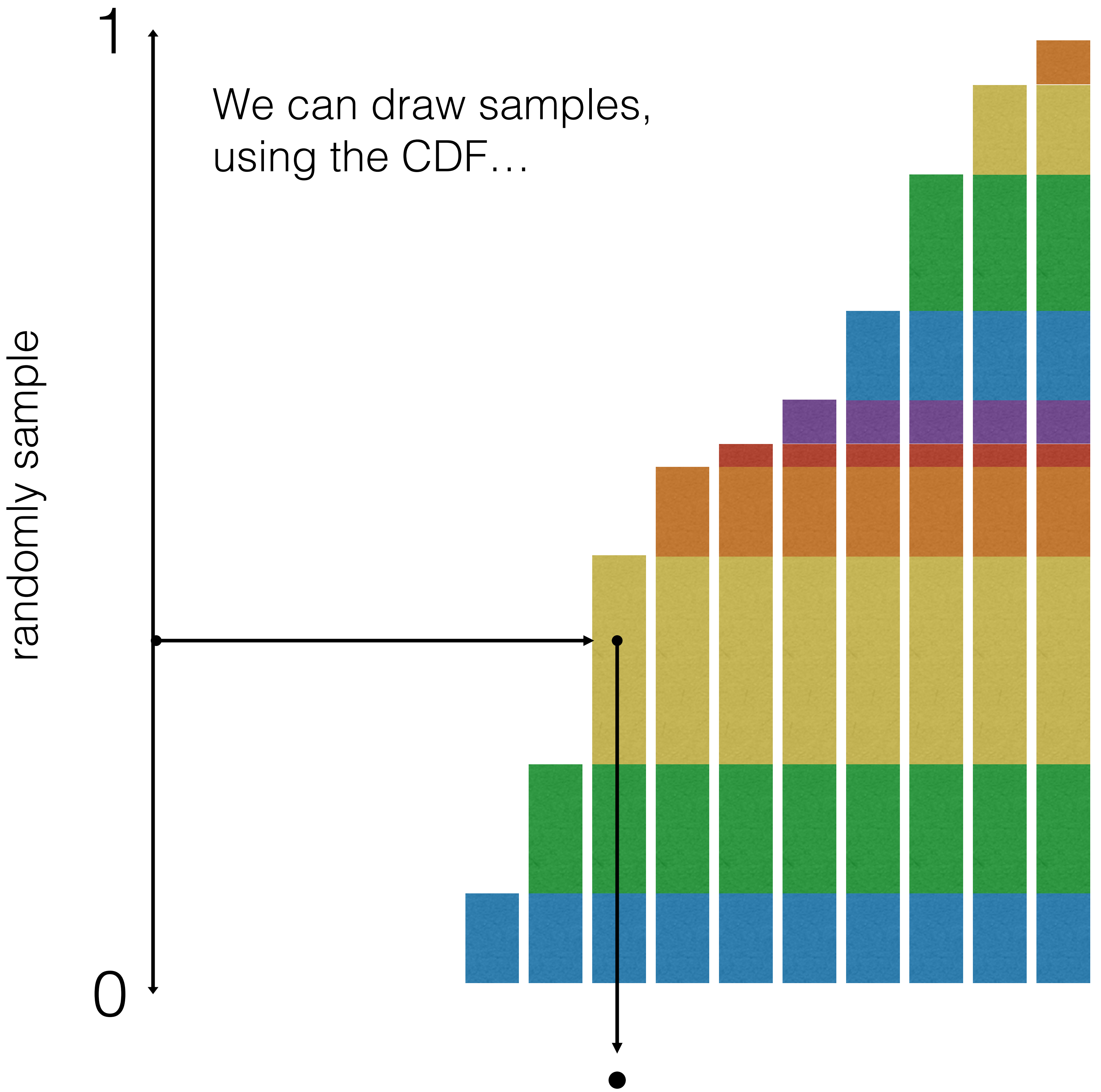


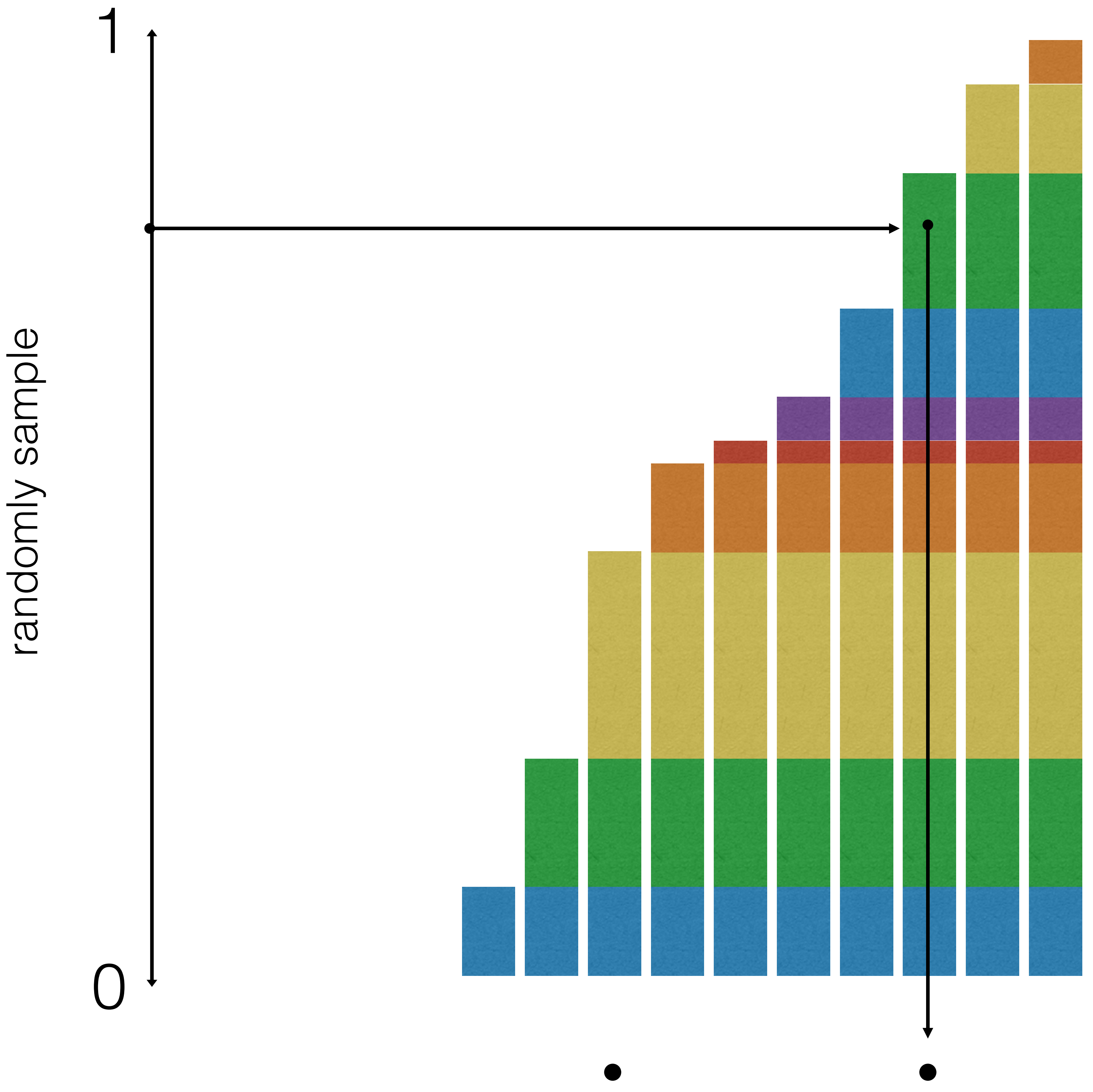
probability density function

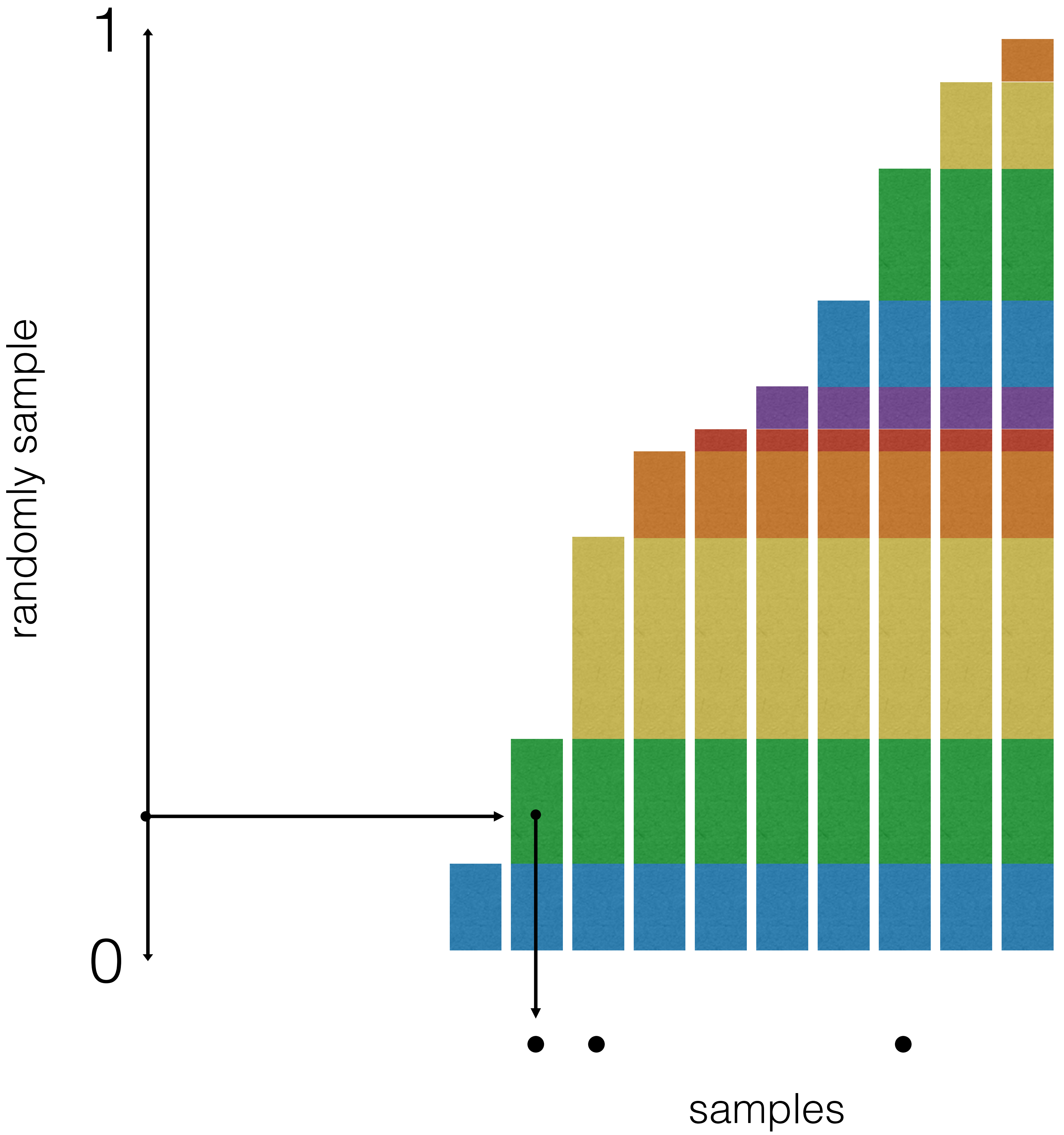


cumulative density function

We can draw samples,
using the CDF…

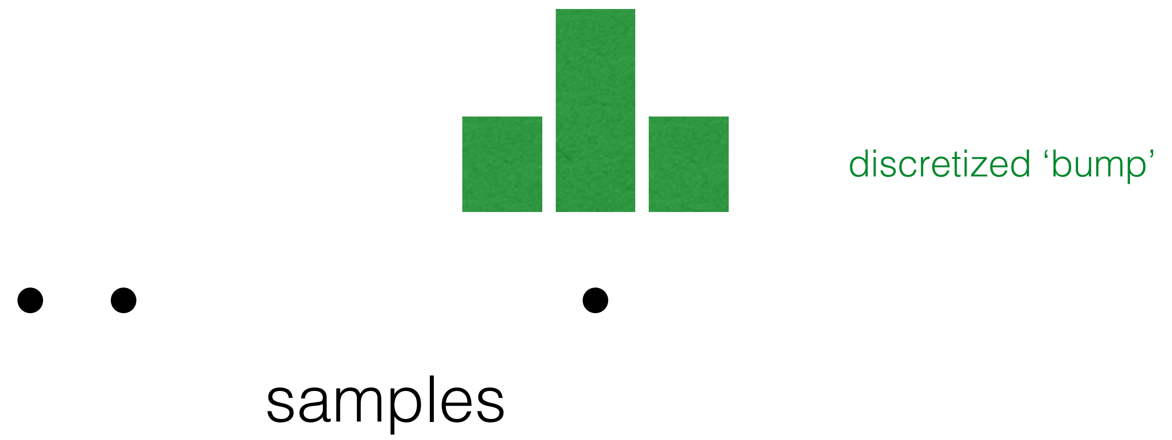



1
randomly sample
0

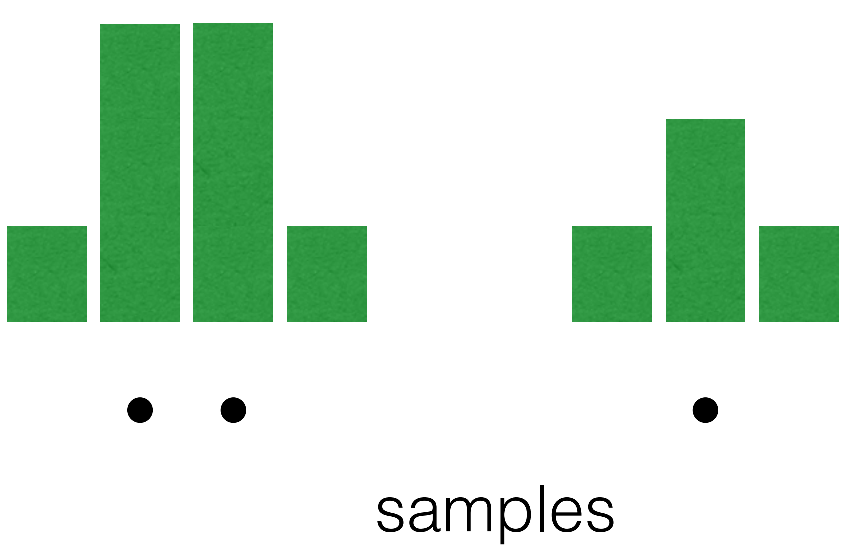

1
0
randomly sample
samples

Now to estimate the ‘hidden’ PDF
place Gaussian bumps on the samples…

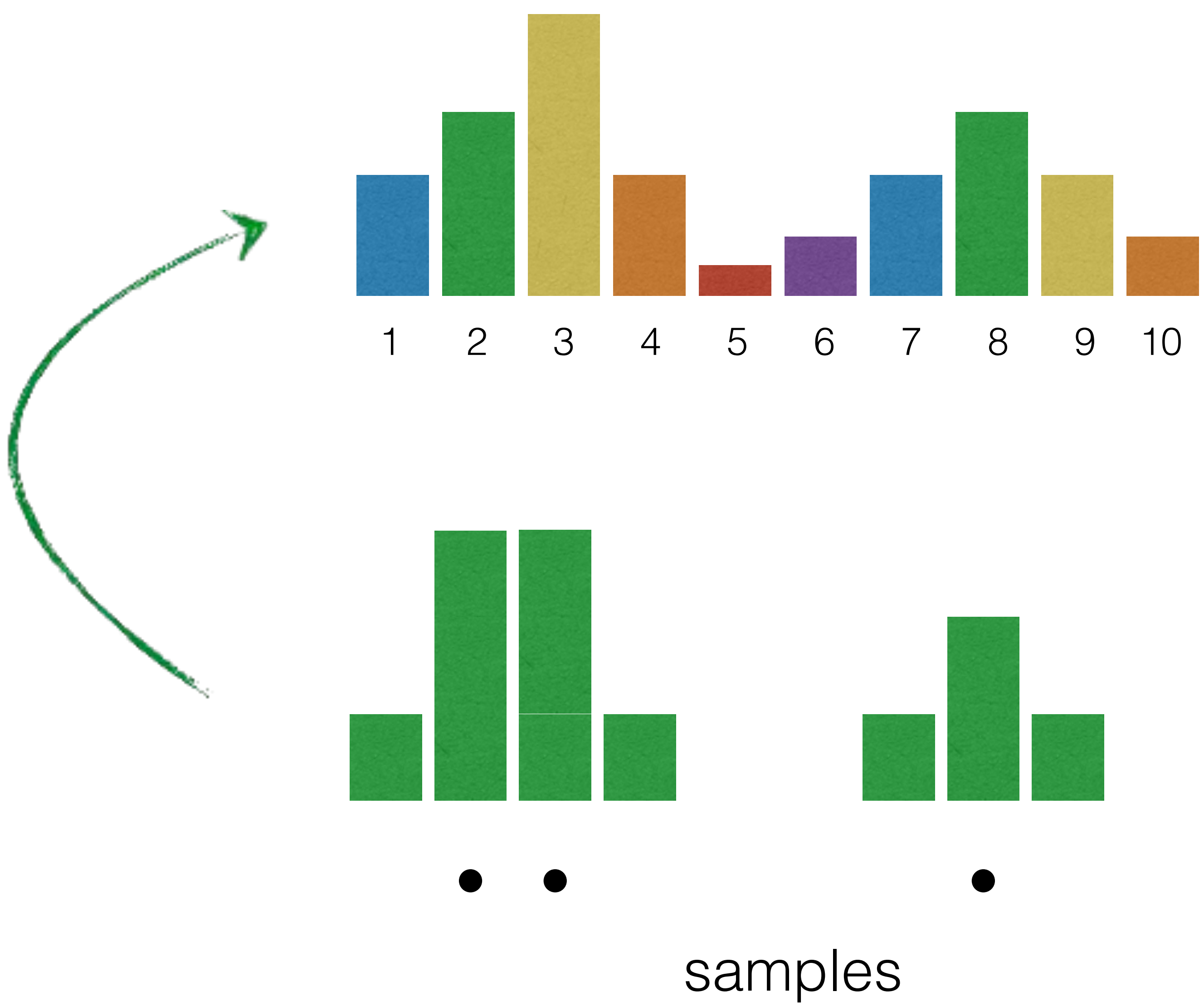
samples


discretized ‘bump’
samples

samples

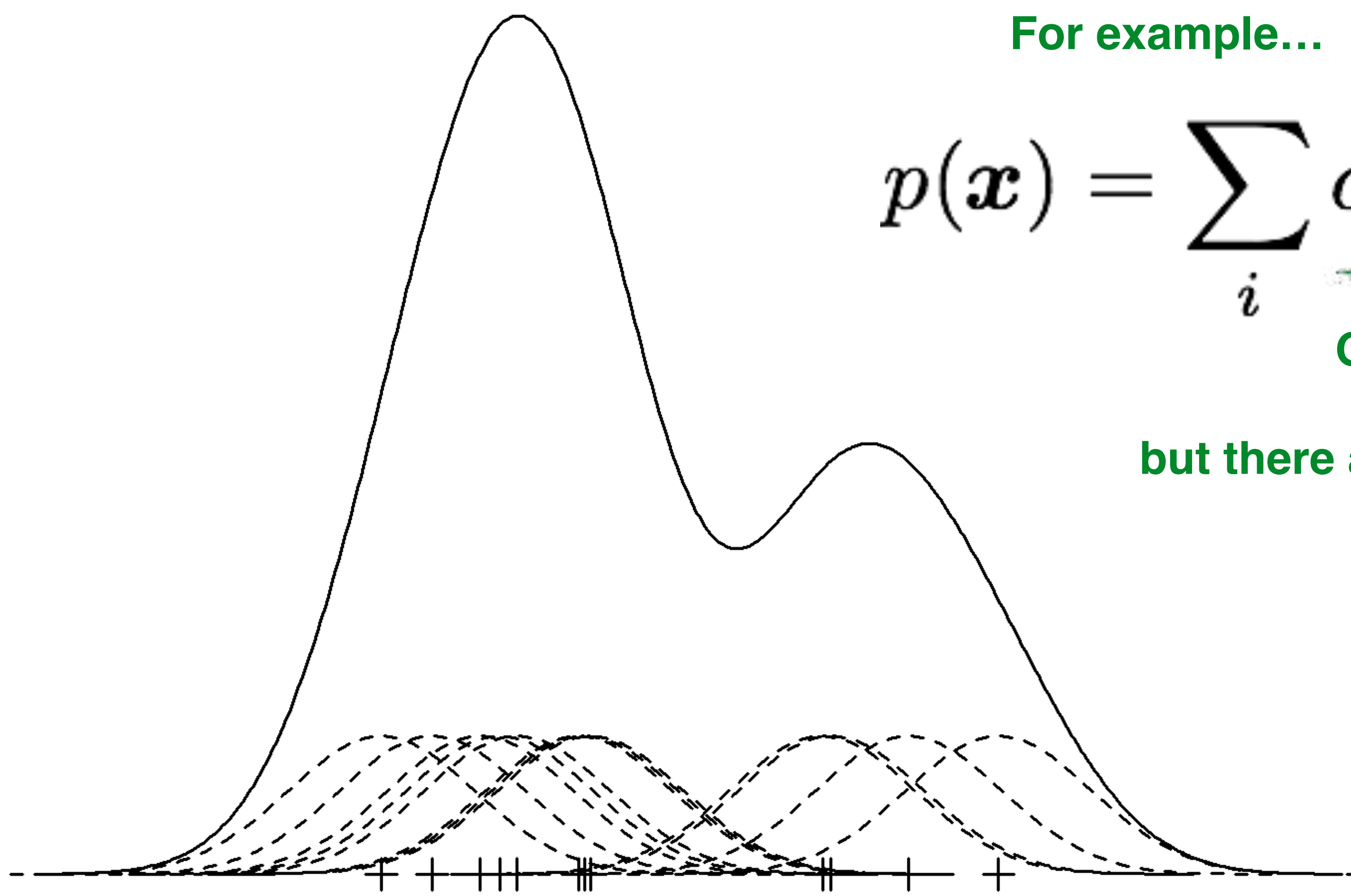

samples

Kernel Density Estimate approximates the original PDF



Kernel Density Estimation

Approximate the underlying PDF from samples from it



For example…

$$p(\boldsymbol{x}) = \sum_i c_i e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}_i)^2}{2\sigma^2}}$$

Gaussian 'bump' aka 'kernel'

but there are many types of kernels!

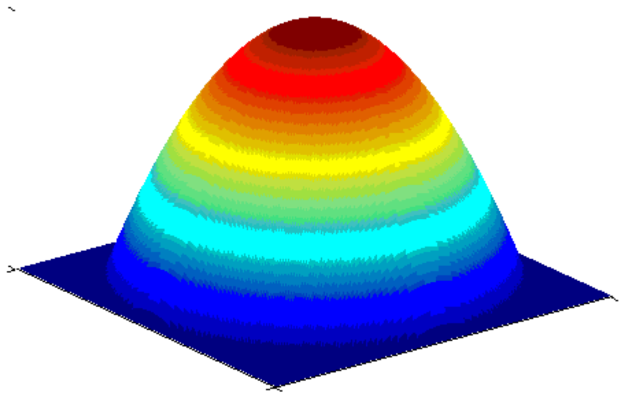
Put 'bump' on every sample to approximate the PDF

Kernel Function

$$K(\boldsymbol{x}, \boldsymbol{x}')$$

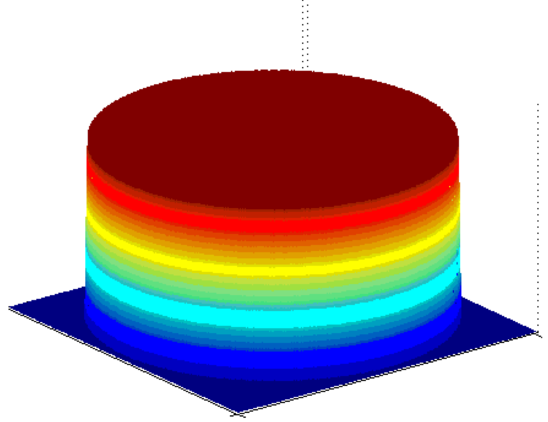
returns the ‘distance’ between two points

Epanechnikov kernel



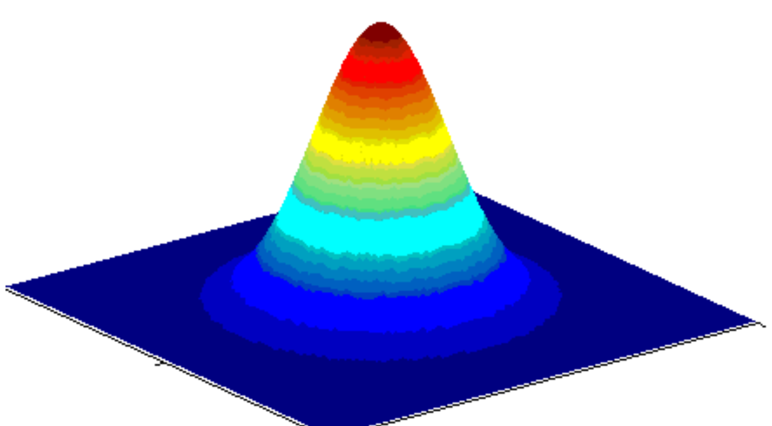
$$K(\boldsymbol{x}, \boldsymbol{x}') = \begin{cases} c(1 - \|\boldsymbol{x} - \boldsymbol{x}'\|^2) & \|\boldsymbol{x} - \boldsymbol{x}'\|^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Uniform kernel



$$K(\boldsymbol{x}, \boldsymbol{x}') = \begin{cases} c & \|\boldsymbol{x} - \boldsymbol{x}'\|^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Normal kernel



$$K(\boldsymbol{x}, \boldsymbol{x}') = c \exp\left(\frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}'\|^2\right)$$

These are all radially symmetric kernels

Radially symmetric kernels

…can be written in terms of its *profile*

$$K(\boldsymbol{x}, \boldsymbol{x}') = c \cdot k(\|\boldsymbol{x} - \boldsymbol{x}'\|^2)$$

profile

Connecting KDE and the Mean Shift Algorithm

Mean-Shift Tracking

Given a set of points:

$$\{\boldsymbol{x}_s\}_{s=1}^{S} \qquad \boldsymbol{x}_s \in \mathcal{R}^d$$

and a kernel:

$$K(\boldsymbol{x}, \boldsymbol{x}')$$

Find the mean sample point:

$$\boldsymbol{x}$$

Mean-Shift Algorithm

`Initialize` $\boldsymbol{x}$ place we start

`While` $v(\boldsymbol{x}) > \epsilon$ shift values becomes really small

1. `Compute mean-shift`

$$m(\boldsymbol{x}) = \frac{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s)\boldsymbol{x}_s}{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s)}$$

compute the 'mean'

$$v(\boldsymbol{x}) = m(\boldsymbol{x}) - \boldsymbol{x}$$

compute the 'shift'

2. `Update` $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x})$ update the point

Where does this algorithm come from?

Mean-Shift Algorithm

Initialize $\boldsymbol{x}$

While $v(\boldsymbol{x}) > \epsilon$

1. Compute mean-shift

$$m(\boldsymbol{x}) = \frac{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s)\boldsymbol{x}_s}{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s)}$$

Where does this come from?

$$v(\boldsymbol{x}) = m(\boldsymbol{x}) - \boldsymbol{x}$$

2. Update $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x})$

Where does this algorithm come from?

How is the KDE related to the mean shift algorithm?

Recall:

Kernel density estimate

(radially symmetric kernels)

$$P(\boldsymbol{x}) = \frac{1}{N} c \sum_{n} k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

can compute probability for any point using the KDE!

We can show that:

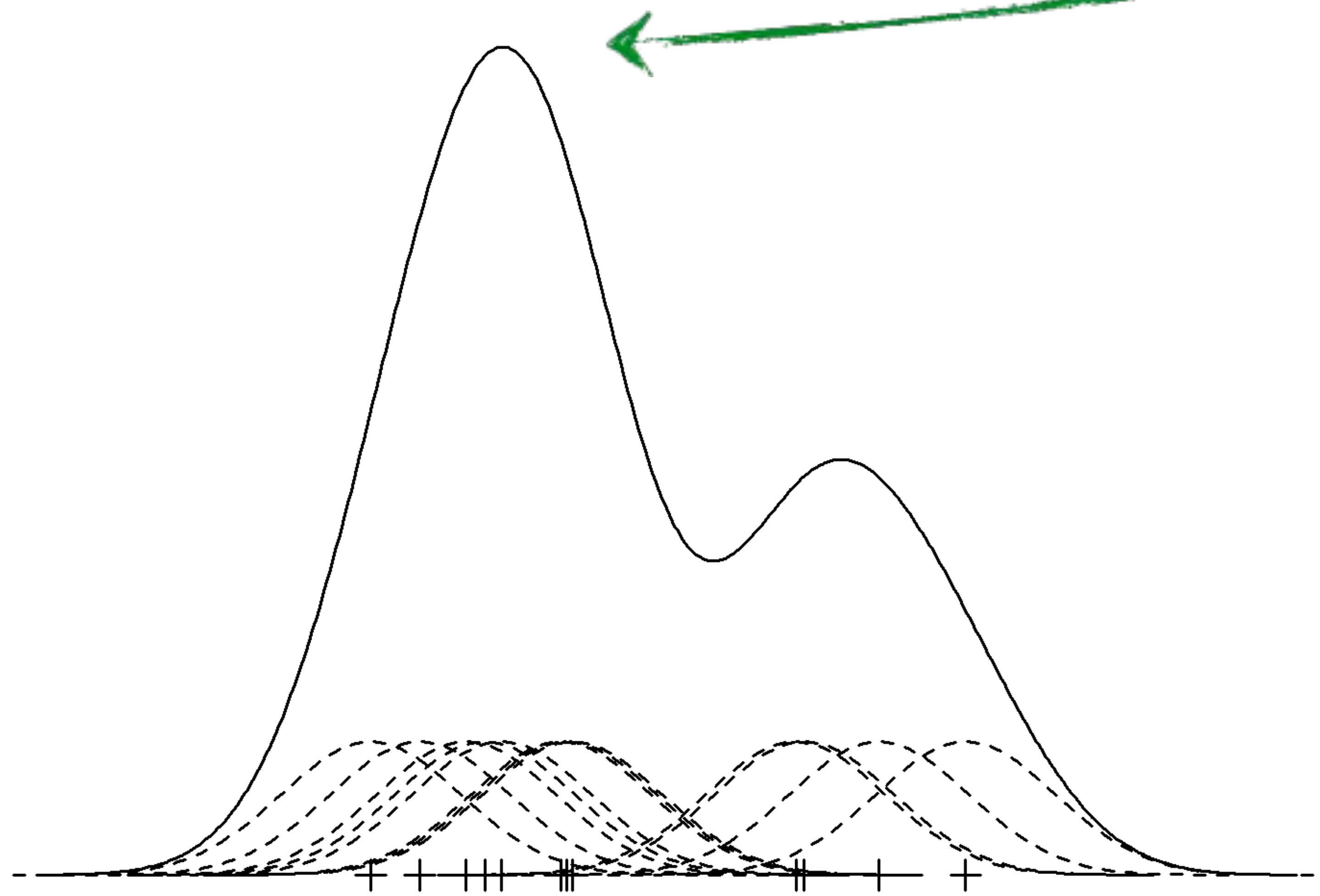
Gradient of the PDF is related to the mean shift vector

$$\nabla P(\boldsymbol{x}) \propto m(\boldsymbol{x})$$

The <u>mean shift vector</u> is a ‘step’ in the direction of the gradient of the KDE

mean-shift algorithm is maximizing the objective function

In mean-shift tracking, we are trying to find this



which means we are trying to…

We are trying to optimize this:

$$\boldsymbol{x} = \arg\max_{\boldsymbol{x}} P(\boldsymbol{x})$$

find the solution that has the highest probability

$$= \arg\max_{\boldsymbol{x}} \frac{1}{N} c \sum_n k(||\boldsymbol{x} - \boldsymbol{x}_n||^2)$$

usually non-linear

non-parametric

How do we optimize this non-linear function?

We are trying to optimize this:

$$\boldsymbol{x} = \arg\max_{\boldsymbol{x}} P(\boldsymbol{x})$$

$$= \arg\max_{\boldsymbol{x}} \frac{1}{N} c \sum_{n} k(||\boldsymbol{x} - \boldsymbol{x}_n||^2)$$

usually non-linear

non-parametric

How do we optimize this non-linear function?

compute partial derivatives … **gradient descent!**

$$P(\boldsymbol{x}) = \frac{1}{N} c \sum_n k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Compute the gradient

$$P(\boldsymbol{x}) = \frac{1}{N} c \sum_n k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} c \sum_n \nabla k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Expand the gradient (algebra)

$$P(\boldsymbol{x}) = \frac{1}{N} c \sum_n k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} c \sum_n \nabla k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Expand gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n (\boldsymbol{x} - \boldsymbol{x}_n) k'(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

$$P(\boldsymbol{x}) = \frac{1}{N} c \sum_n k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} c \sum_n \nabla k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Expand gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n (\boldsymbol{x} - \boldsymbol{x}_n) k'(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Call the gradient of the kernel function g

$$k'(\cdot) = -g(\cdot)$$

$$P(\boldsymbol{x}) = \frac{1}{N} c \sum_n k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} c \sum_n \nabla k(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

Expand gradient

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n (\boldsymbol{x} - \boldsymbol{x}_n) k'(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

change of notation
(kernel-shadow pairs)

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n (\boldsymbol{x}_n - \boldsymbol{x}) g(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

keep this in memory: $k'(\cdot) = -g(\cdot)$

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n (\boldsymbol{x}_n - \boldsymbol{x}) g(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

multiply it out

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n \boldsymbol{x}_n g(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2) - \frac{1}{N} 2c \sum_n \boldsymbol{x} g(\|\boldsymbol{x} - \boldsymbol{x}_n\|^2)$$

too long!
(use short hand notation)

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n \boldsymbol{x}_n g_n - \frac{1}{N} 2c \sum_n \boldsymbol{x} g_n$$

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n \boldsymbol{x}_n g_n - \frac{1}{N} 2c \sum_n \boldsymbol{x} g_n$$

multiply by one!

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n \boldsymbol{x}_n g_n \left(\frac{\sum_n g_n}{\sum_n g_n} \right) - \frac{1}{N} 2c \sum_n \boldsymbol{x} g_n$$

collecting like terms…

$$\nabla P(\boldsymbol{x}) = \frac{1}{N} 2c \sum_n g_n \left(\frac{\sum_n \boldsymbol{x}_n g_n}{\sum_n g_n} - \boldsymbol{x} \right)$$

What's happening here?

mean shift

$$\nabla P(\boldsymbol{x}) = \underbrace{\frac{1}{N}2c\sum_n g_n}_{\text{constant}} \underbrace{\left(\frac{\sum_n \boldsymbol{x}_n g_n}{\sum_n g_n} - \boldsymbol{x}\right)}_{\text{mean shift!}}$$

The **mean shift** is a 'step' in the direction of the gradient of the KDE

Let $$\boldsymbol{v}(\boldsymbol{x}) = \left(\frac{\sum_n \boldsymbol{x}_n g_n}{\sum_n g_n} - \boldsymbol{x}\right) = \frac{\nabla P(\boldsymbol{x})}{\frac{1}{N}2c\sum_n g_n}$$

Can interpret this to be
gradient ascent with
data dependent step size

Mean-Shift Algorithm

Initialize $\boldsymbol{x}$

While $v(\boldsymbol{x}) > \epsilon$

1. Compute mean-shift

$$m(\boldsymbol{x}) = \frac{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s)\boldsymbol{x}_s}{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s)}$$

$$v(\boldsymbol{x}) = m(\boldsymbol{x}) - \boldsymbol{x}$$

2. Update $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x})$

gradient with
adaptive step size

$$\frac{\nabla P(\boldsymbol{x})}{\frac{1}{N} 2c \sum_n g_n}$$

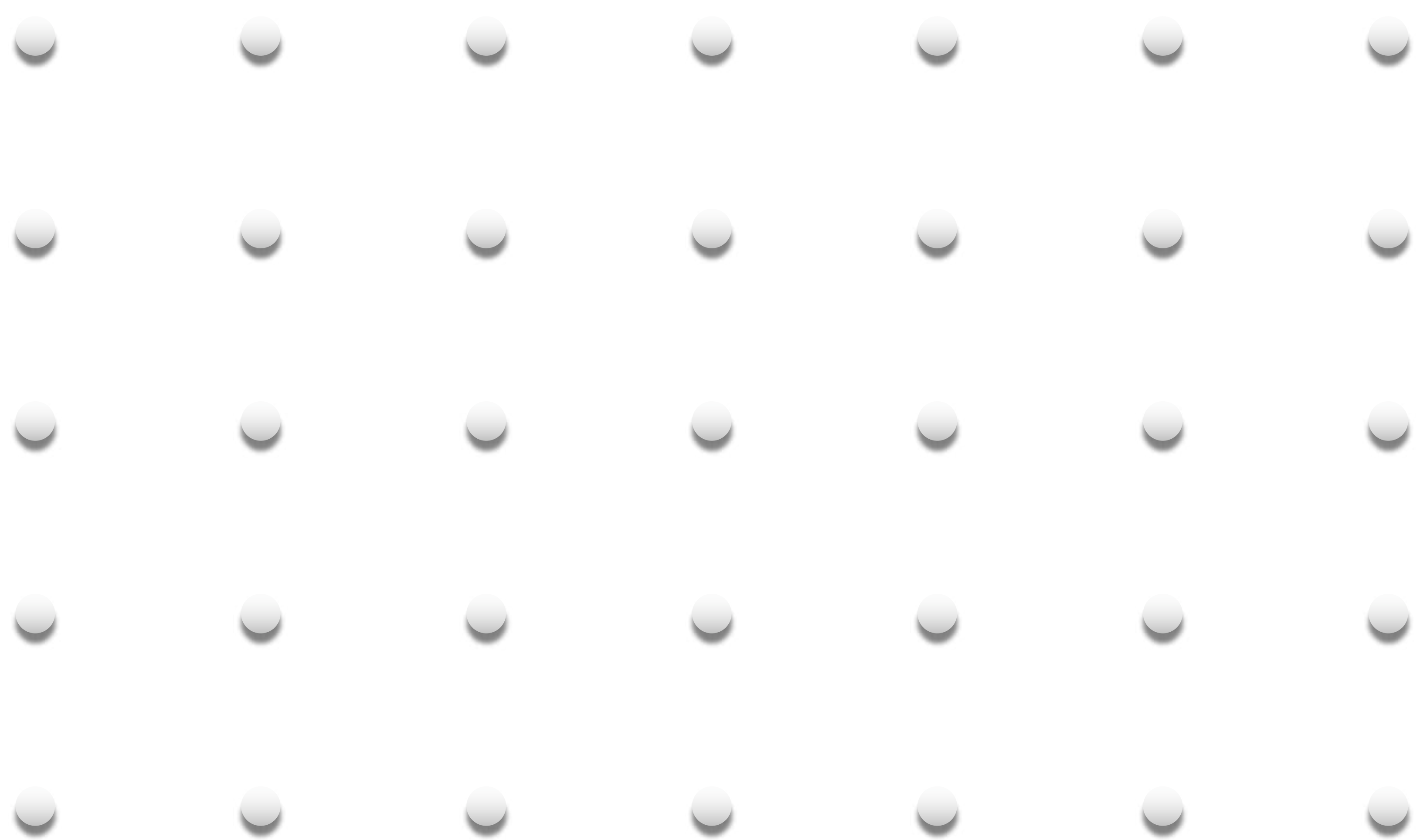
Just 5 lines of code!

Everything up to now has been about distributions over samples…

Mean-shift tracker

Dealing with images

Pixels for a lattice, spatial density is the same everywhere!



What can we do?

same

Consider a set of points: $\{\boldsymbol{x}_s\}_{s=1}^{S}$ $\quad \boldsymbol{x}_s \in \mathcal{R}^d$

Associated weights: $w(\boldsymbol{x}_s)$

Sample mean: $$m(\boldsymbol{x}) = \frac{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s) w(\boldsymbol{x}_s) \boldsymbol{x}_s}{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s) w(\boldsymbol{x}_s)}$$

same

Mean shift: $m(\boldsymbol{x}) - \boldsymbol{x}$

Mean-Shift Algorithm

(for images)

Initialize $\boldsymbol{x}$

While $v(\boldsymbol{x}) > \epsilon$

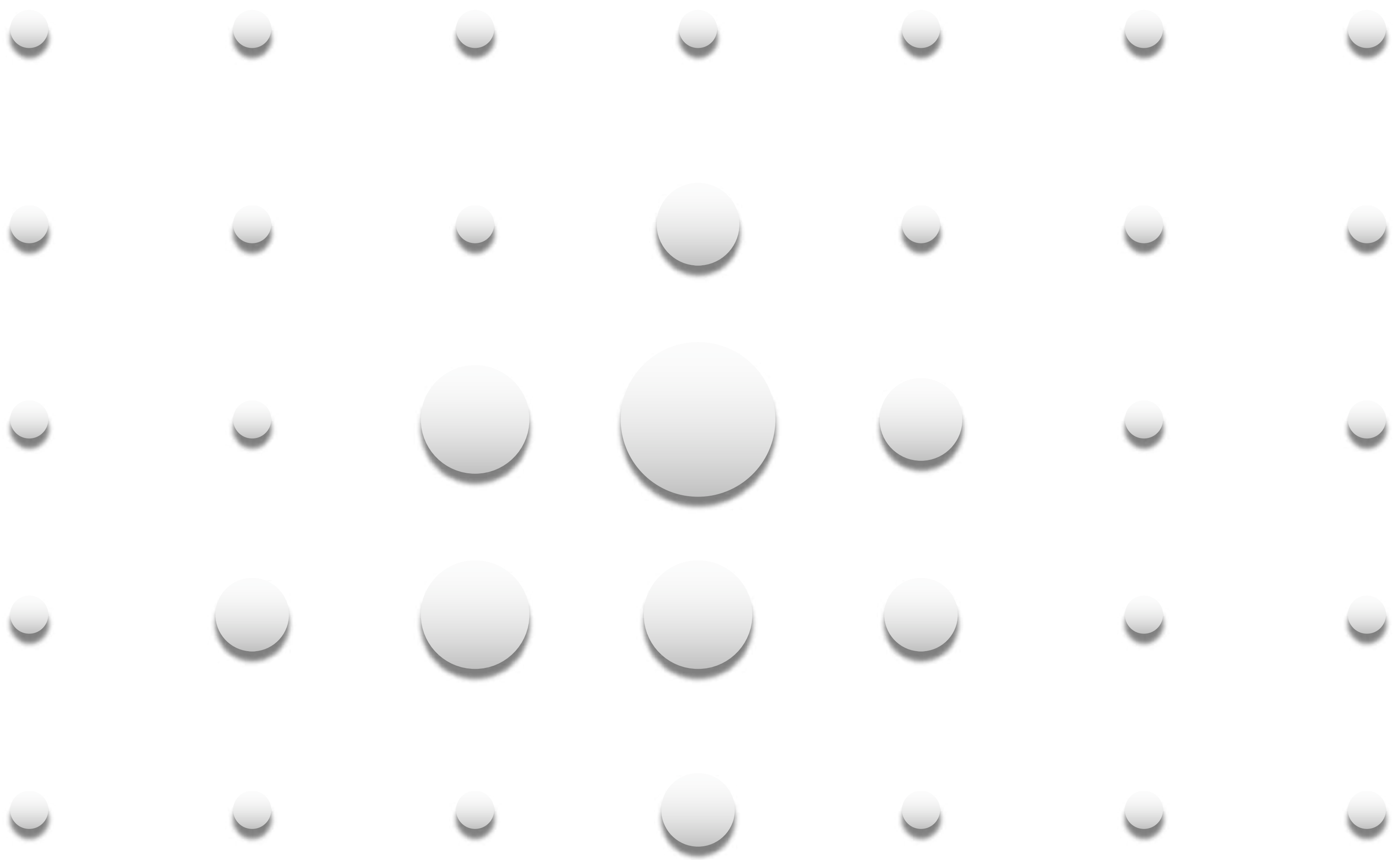
1. Compute mean-shift

$$m(\boldsymbol{x}) = \frac{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s) w(\boldsymbol{x}_s) \boldsymbol{x}_s}{\sum_s K(\boldsymbol{x}, \boldsymbol{x}_s) w(\boldsymbol{x}_s)}$$

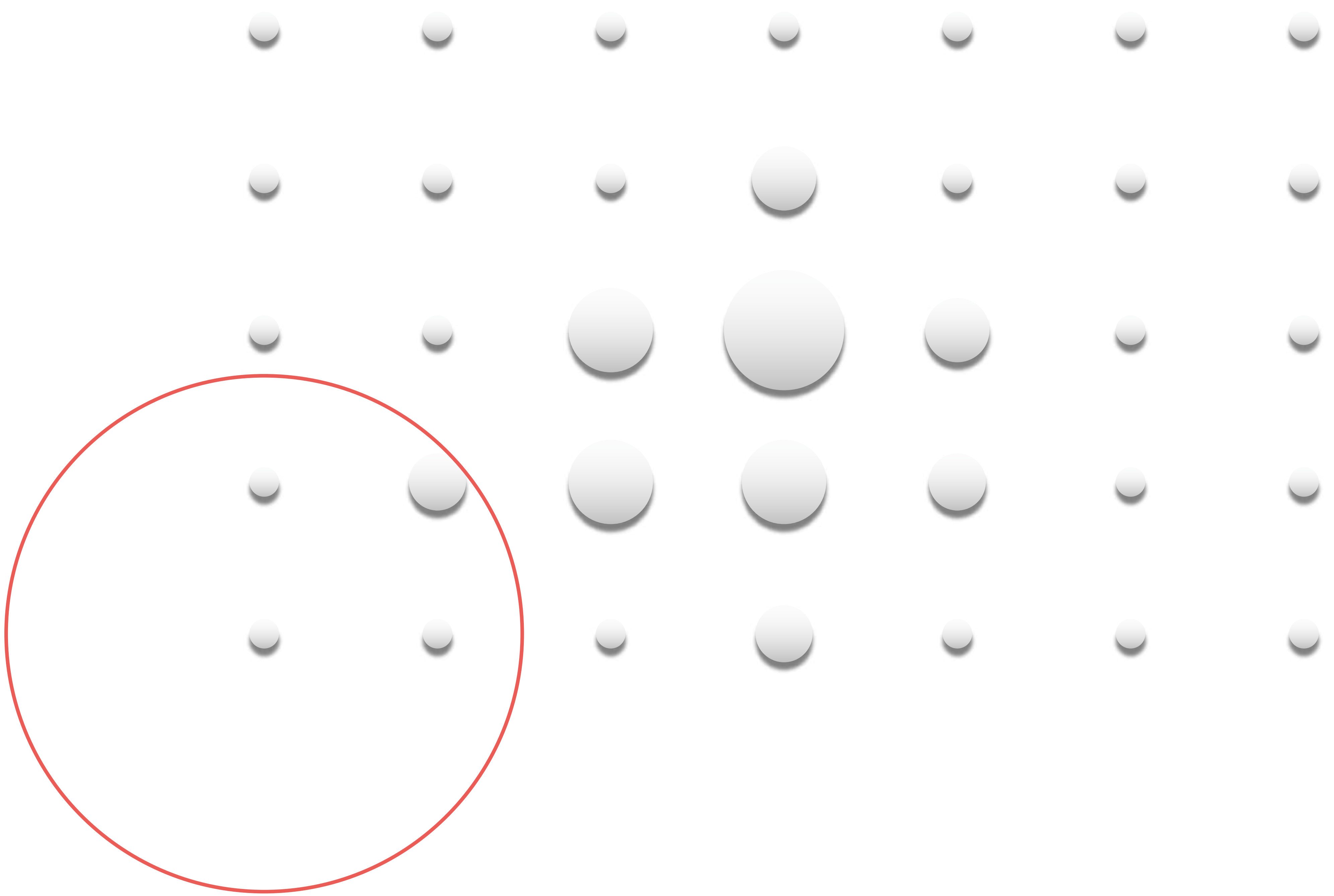
$$v(\boldsymbol{x}) = m(\boldsymbol{x}) - \boldsymbol{x}$$

2. Update $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x})$

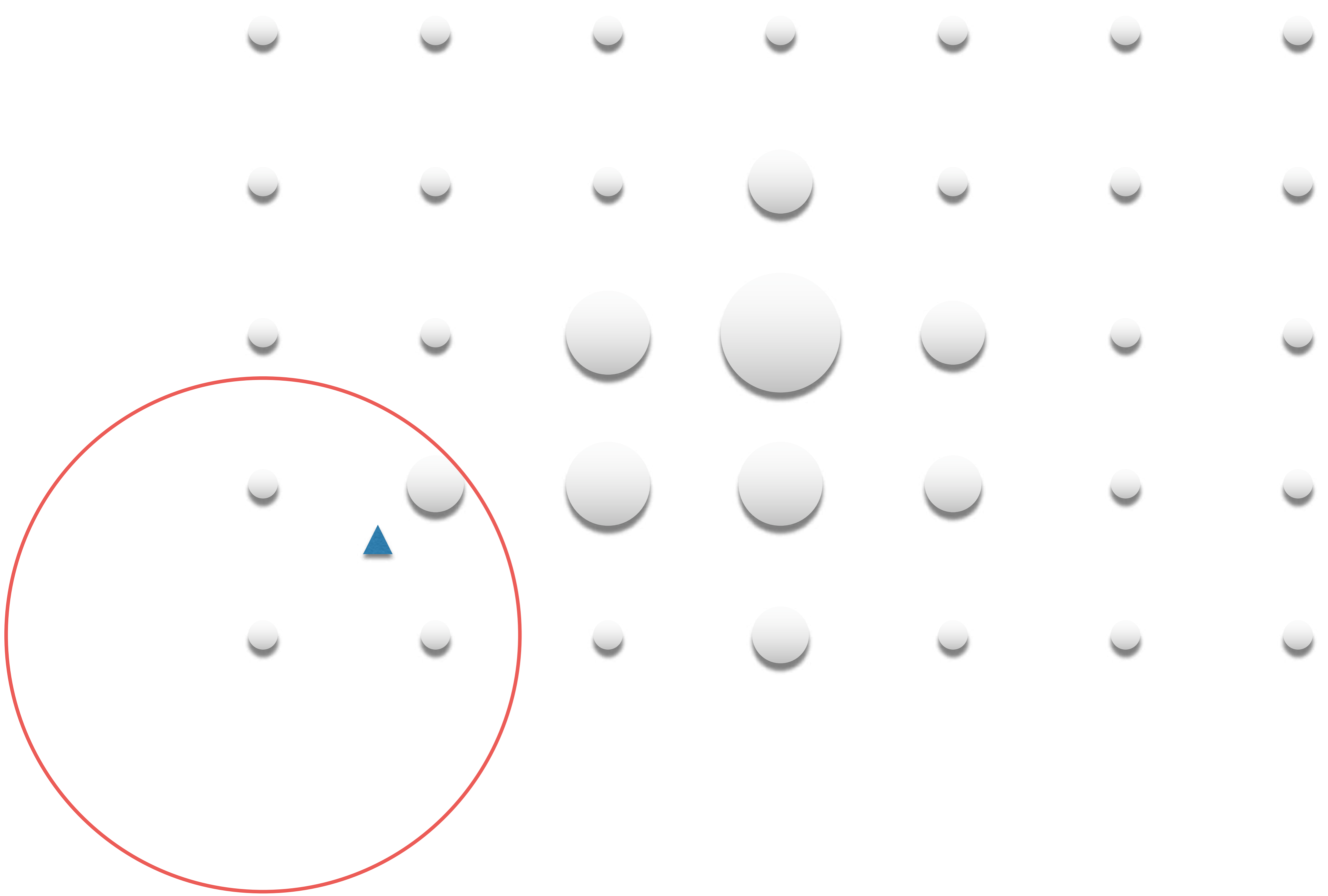
For images, each pixel is point with a weight



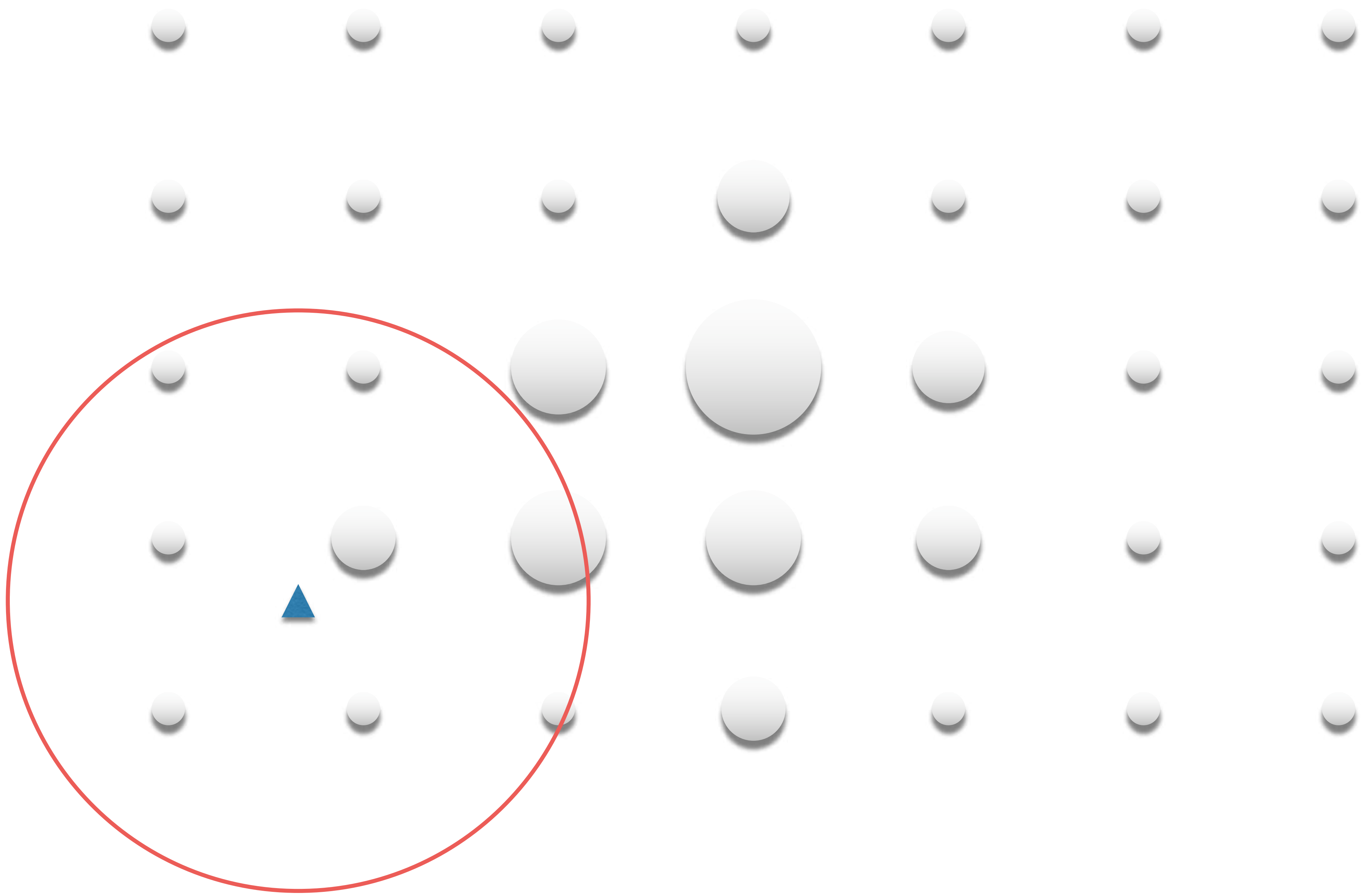
For images, each pixel is point with a weight



For images, each pixel is point with a weight



For images, each pixel is point with a weight



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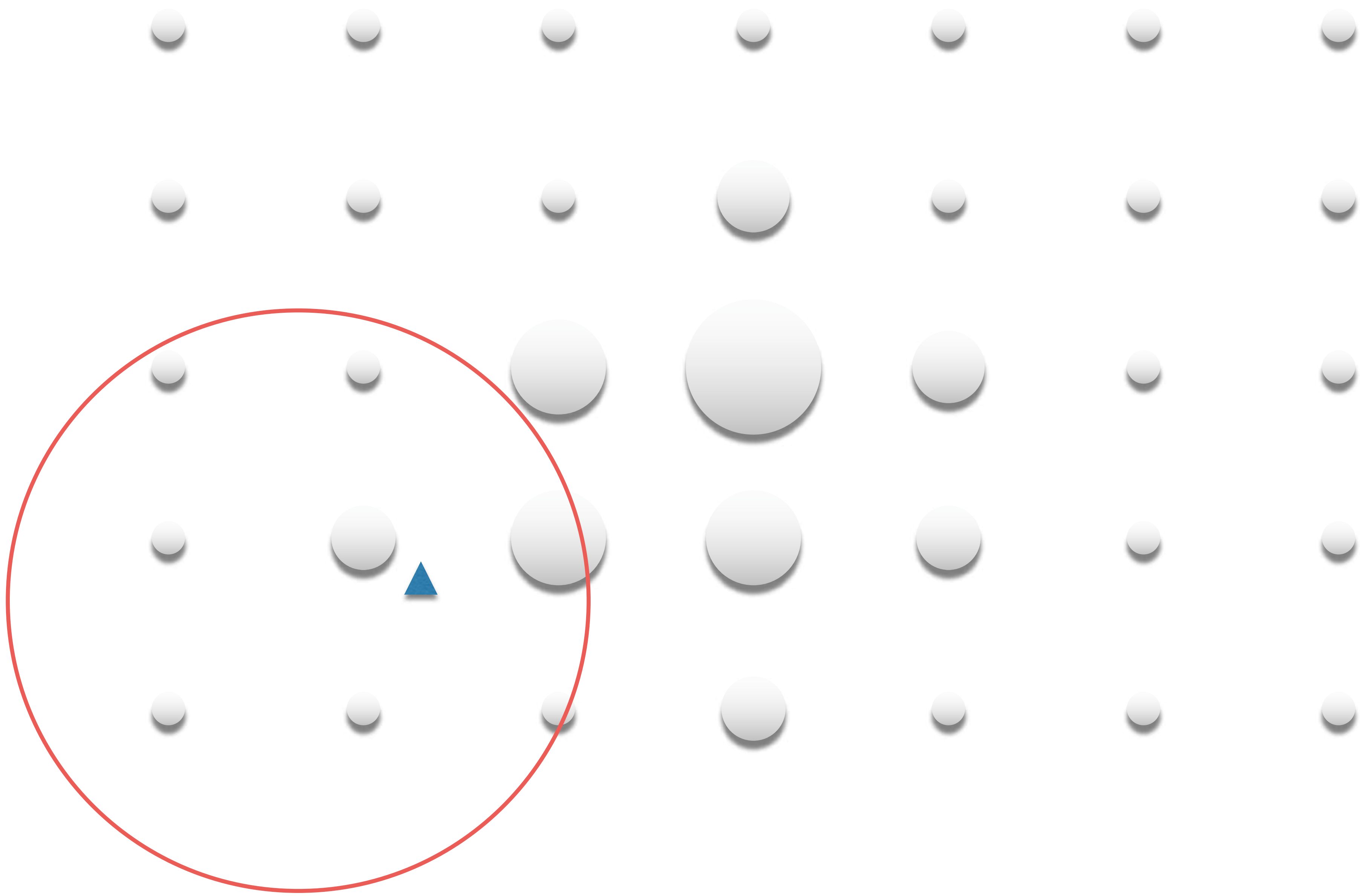

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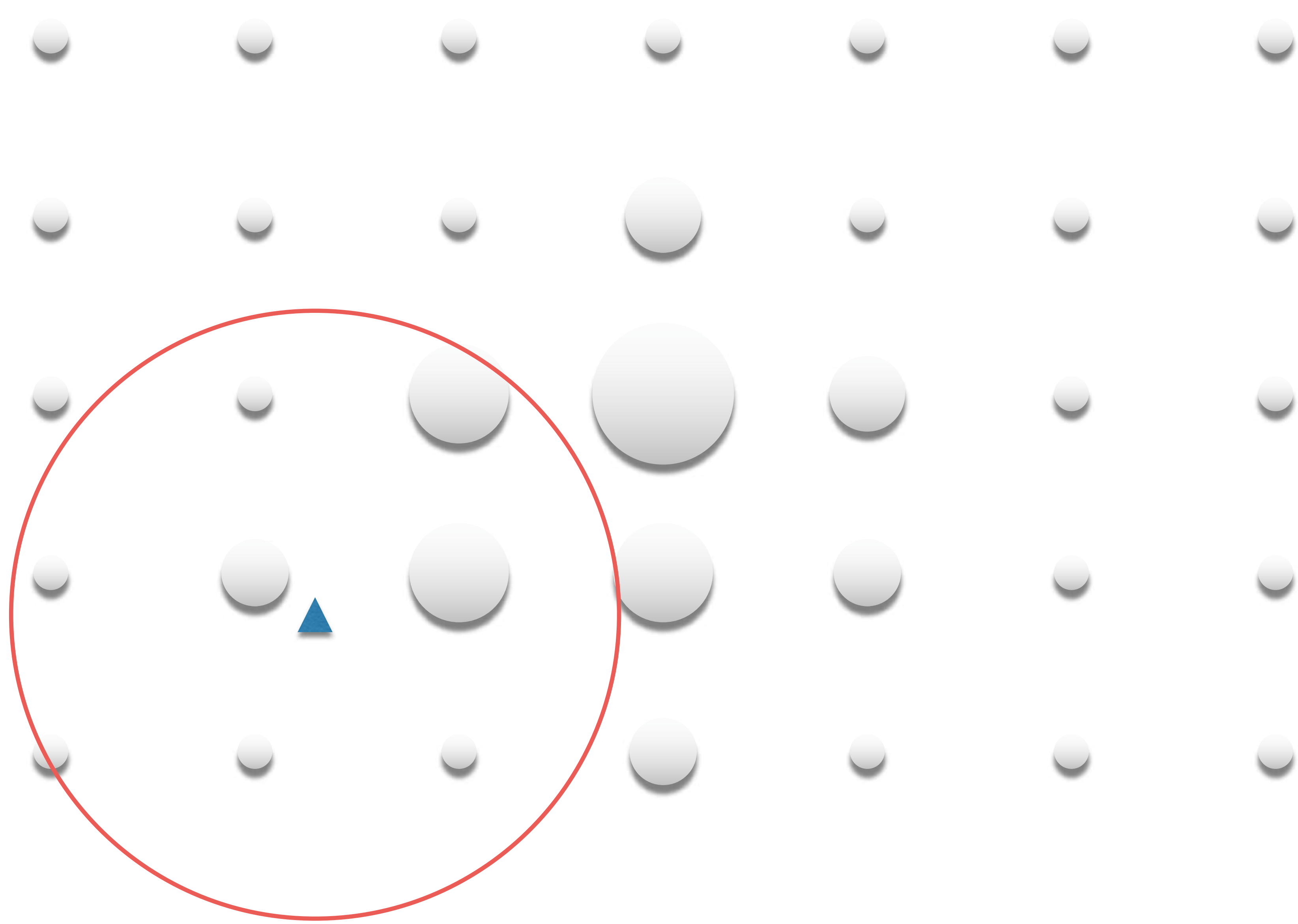
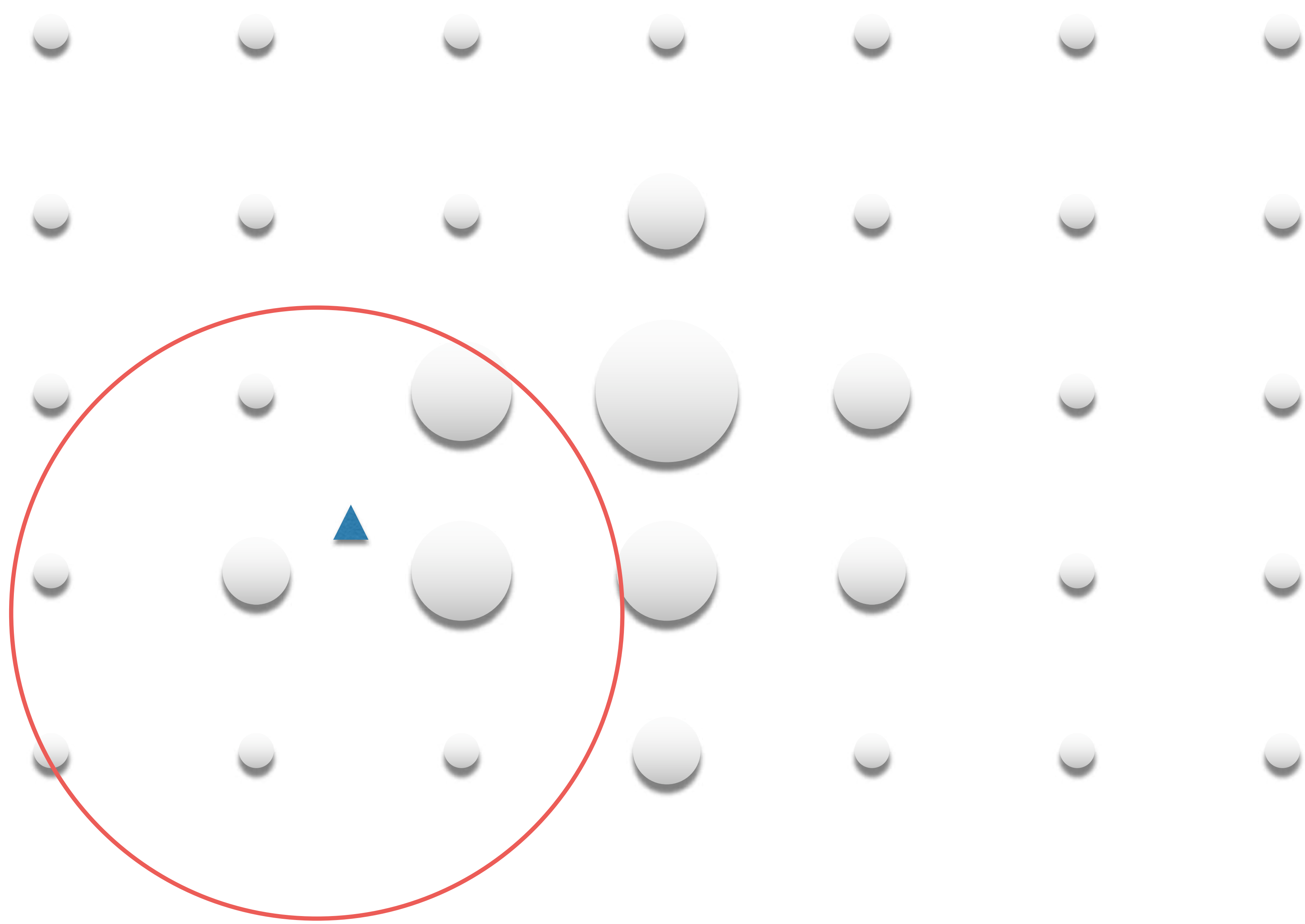
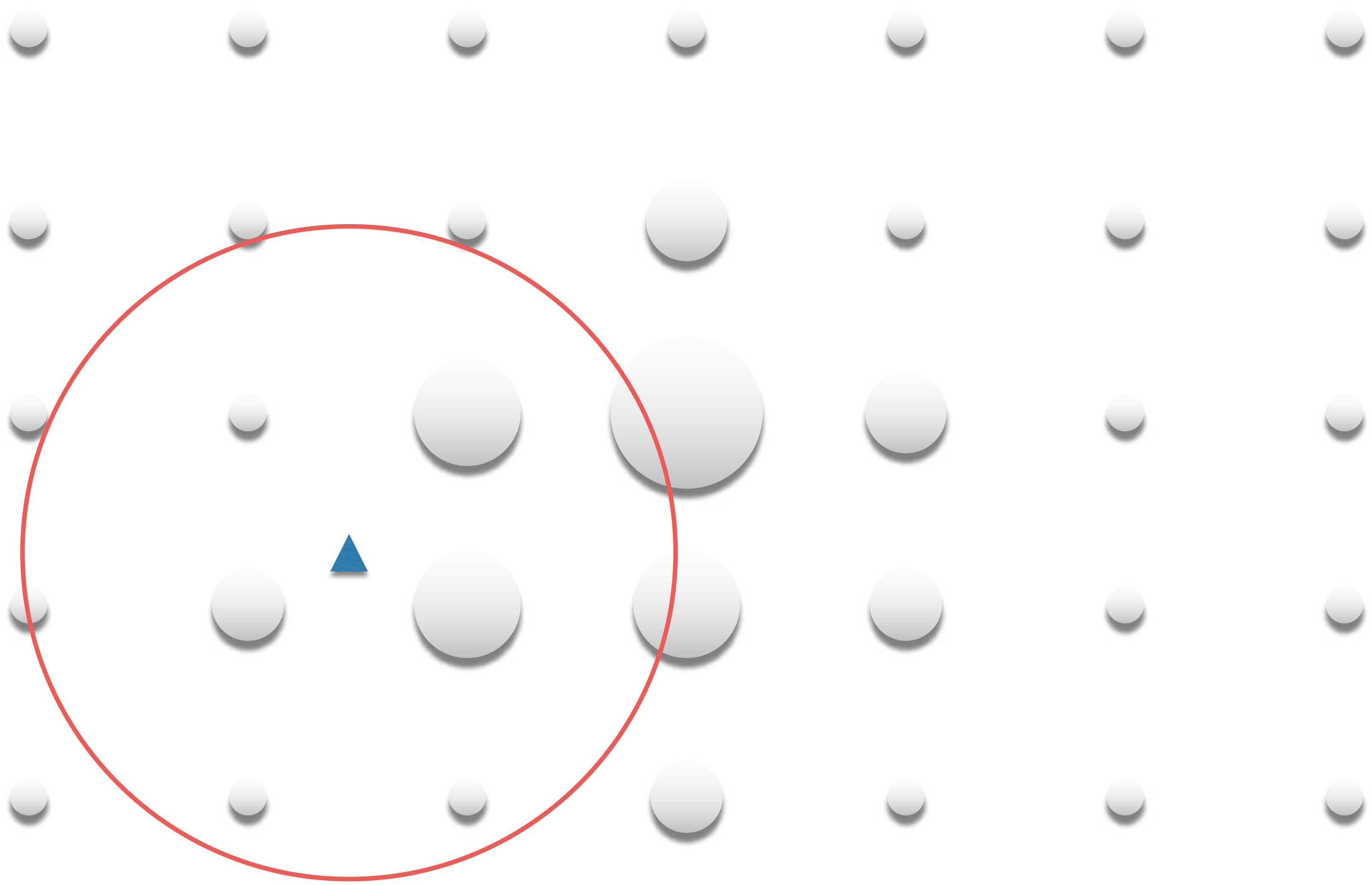


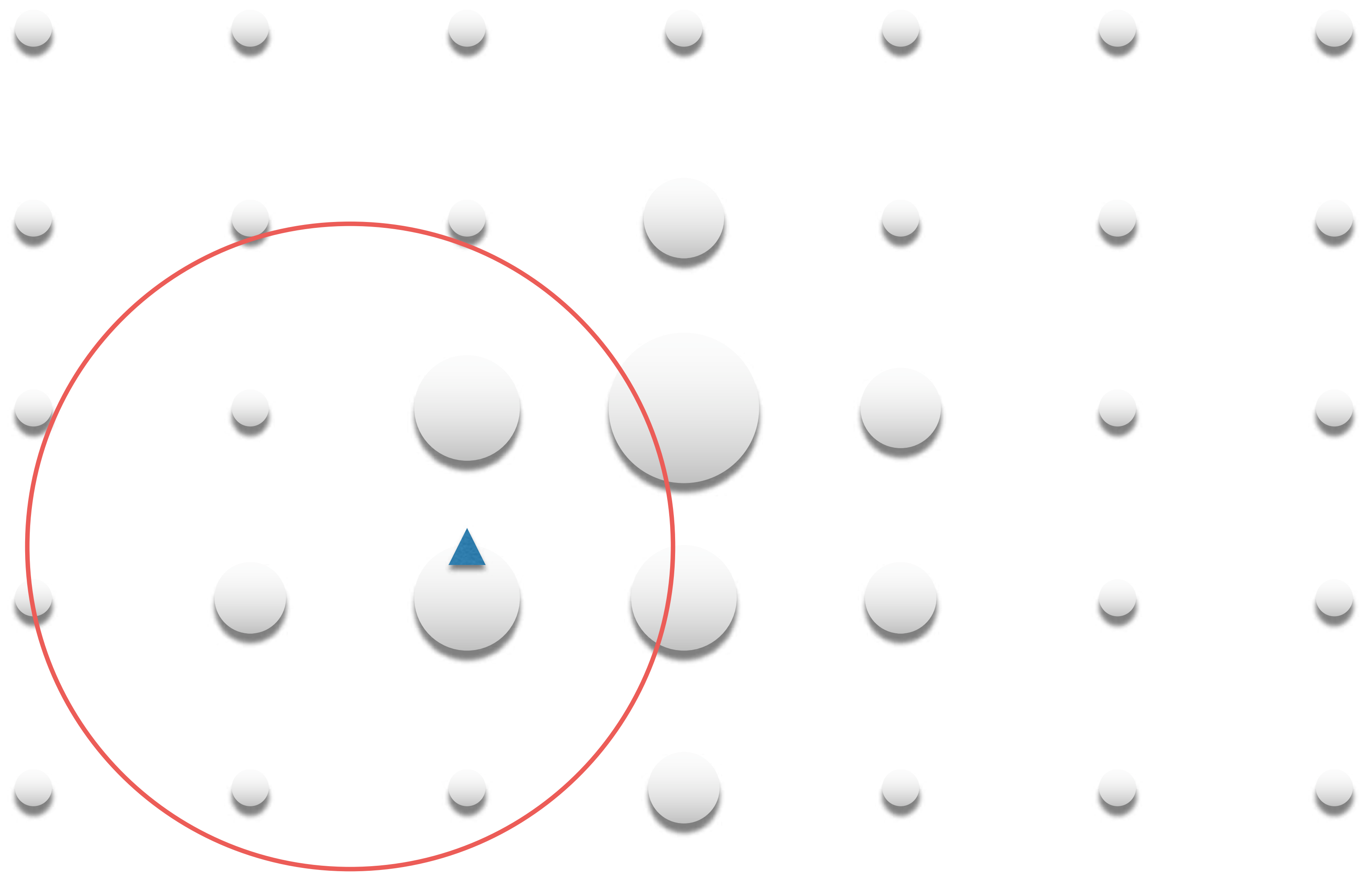
For images, each pixel is point with a weight



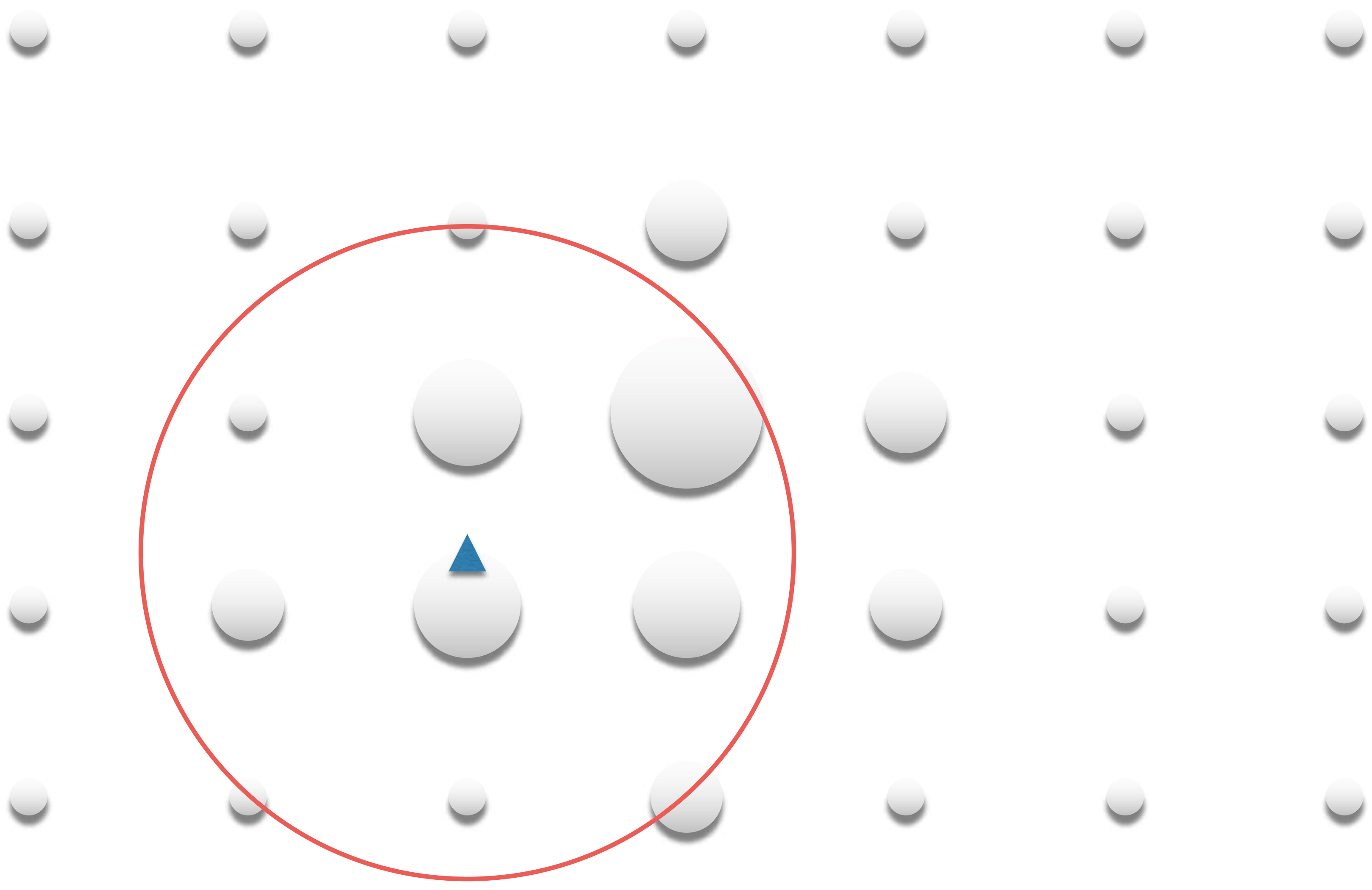
For images, each pixel is point with a weight



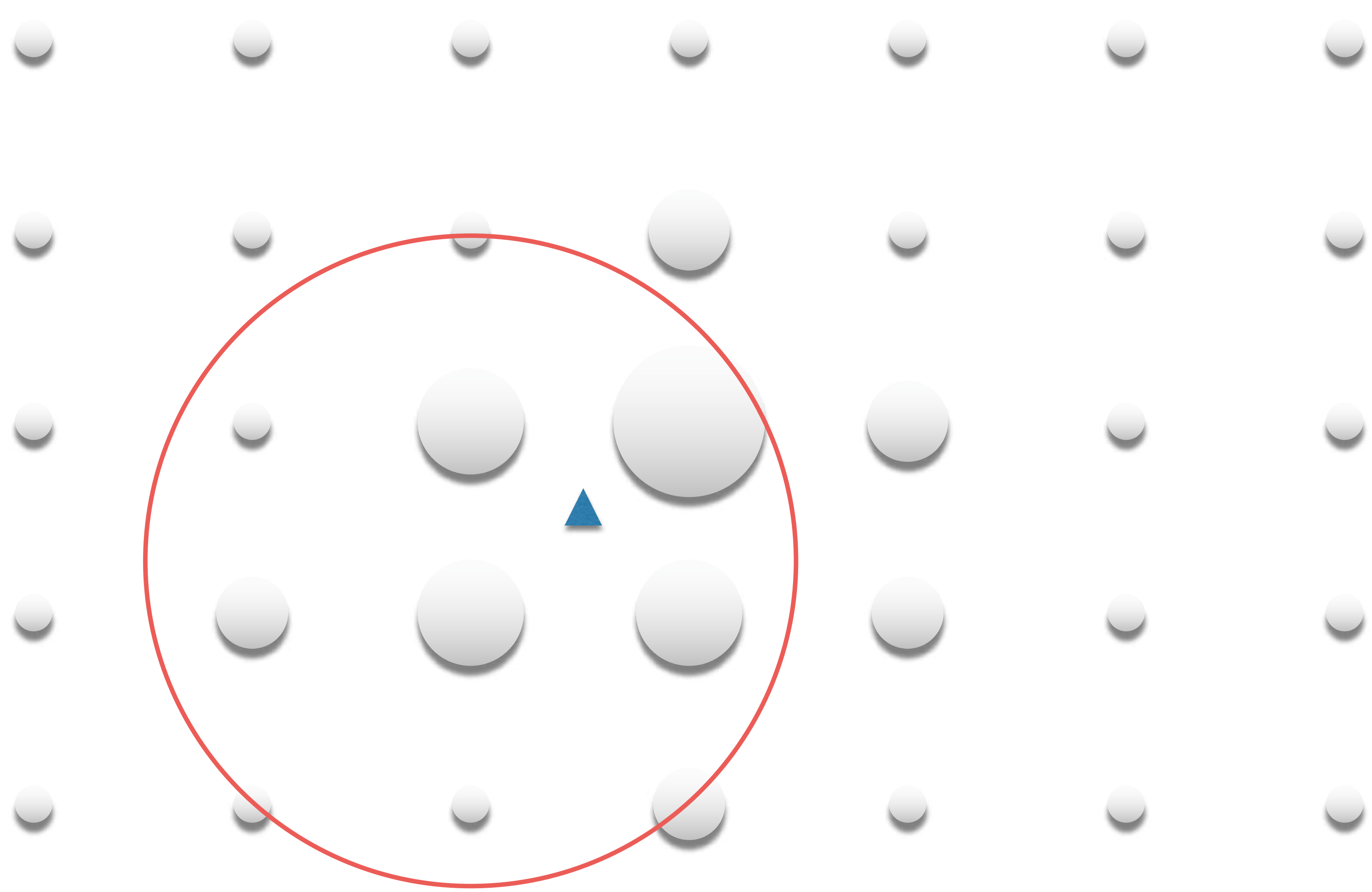
For images, each pixel is point with a weight



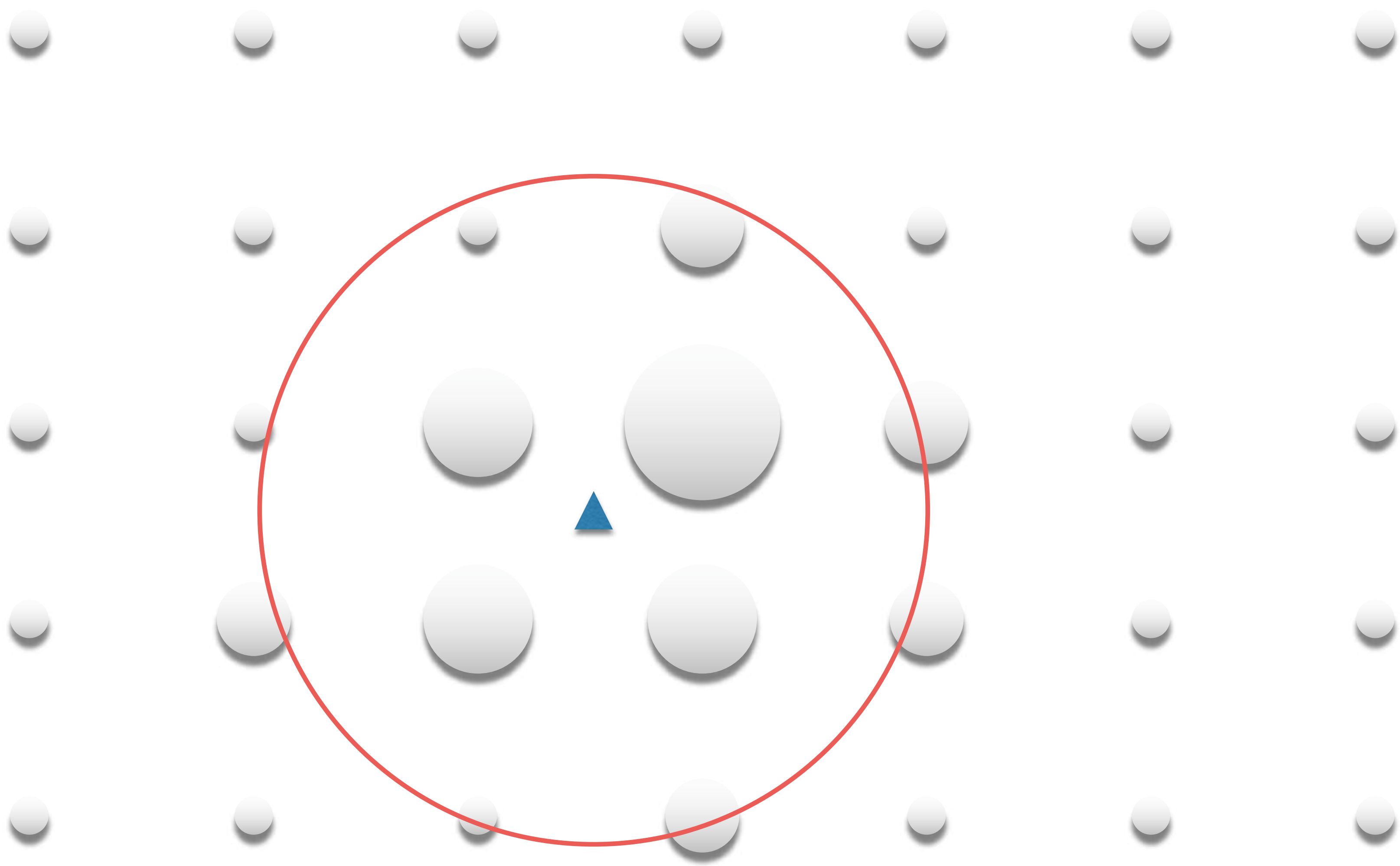
For images, each pixel is point with a weight



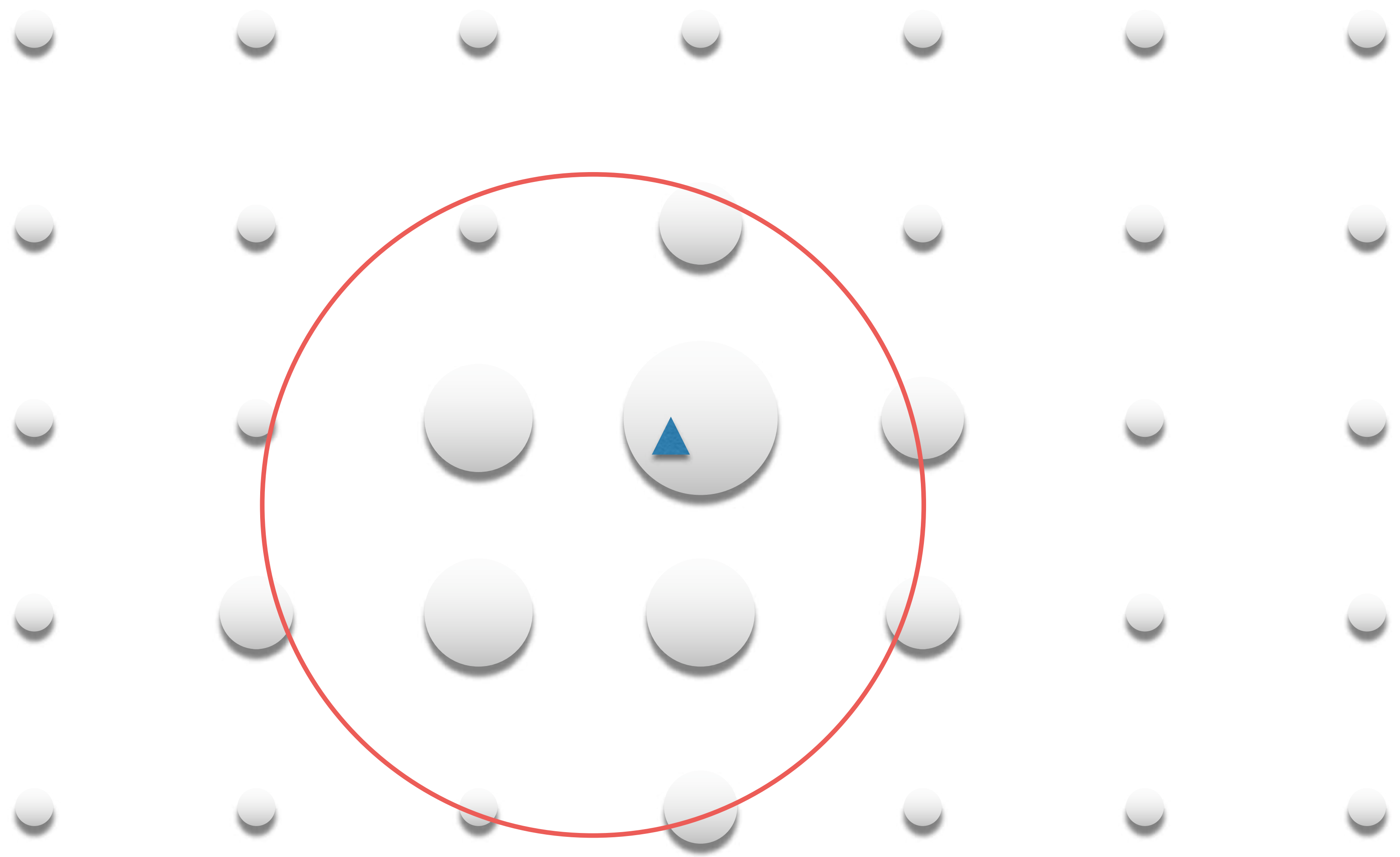
For images, each pixel is point with a weight



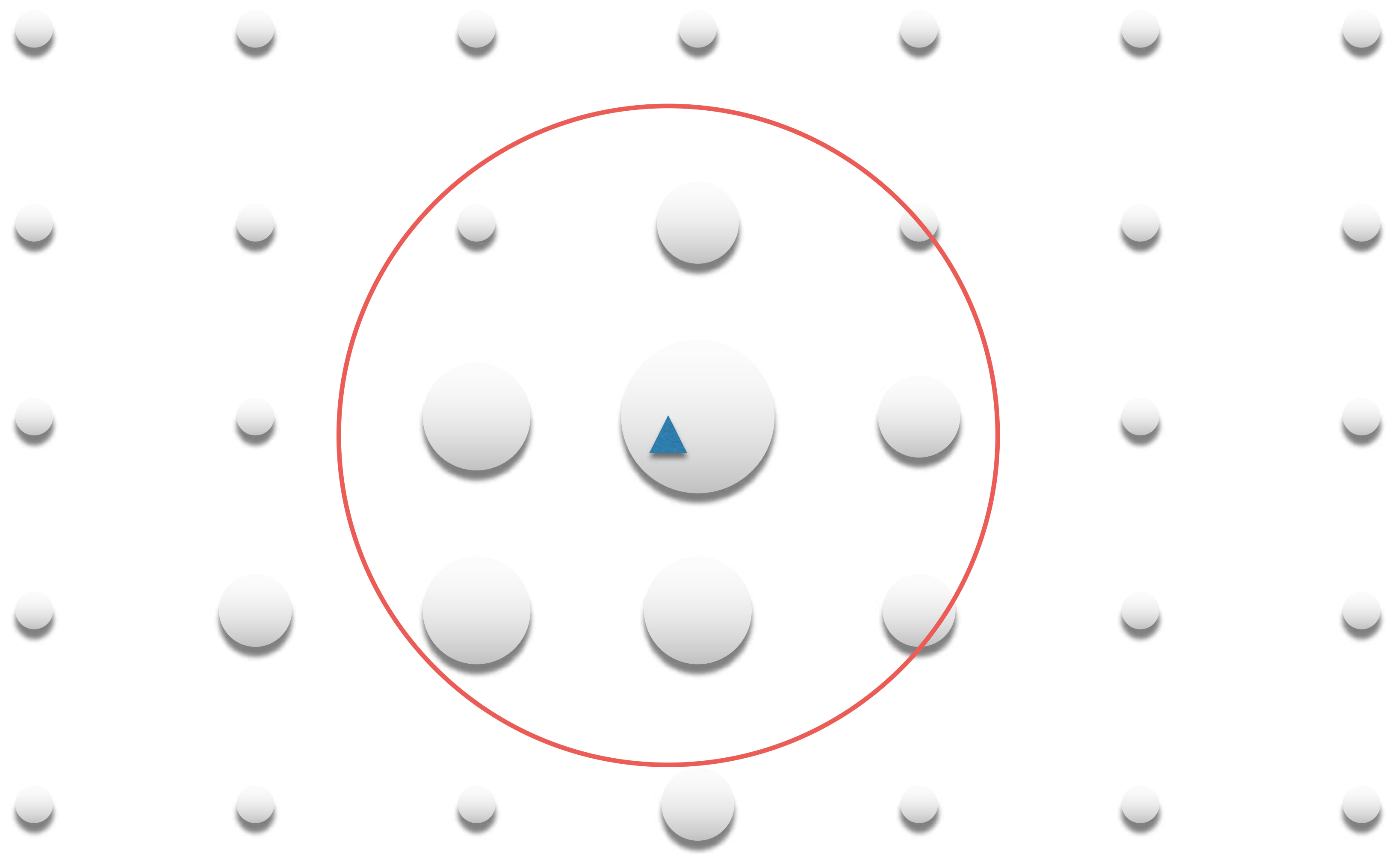
For images, each pixel is point with a weight



For images, each pixel is point with a weight

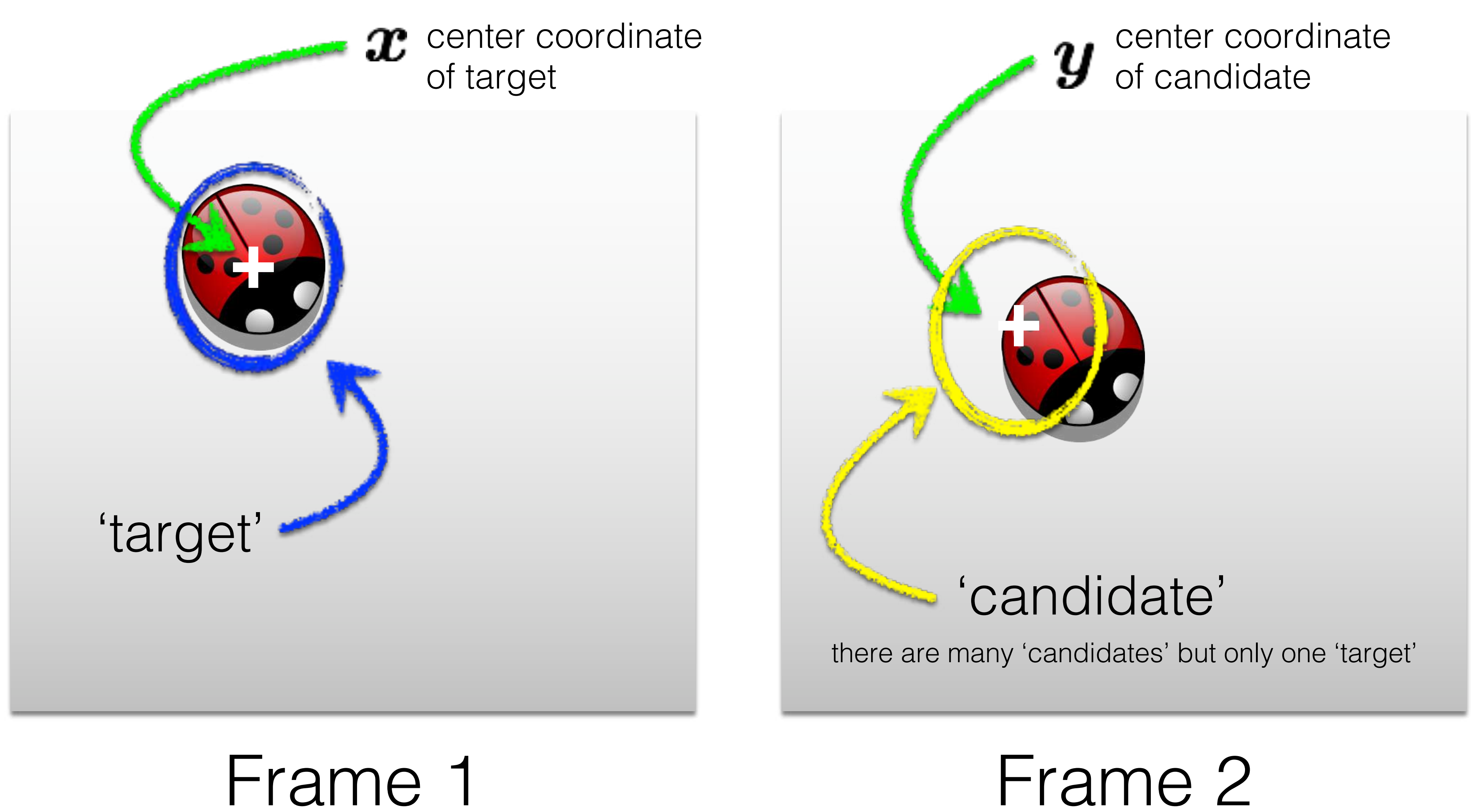


For images, each pixel is point with a weight



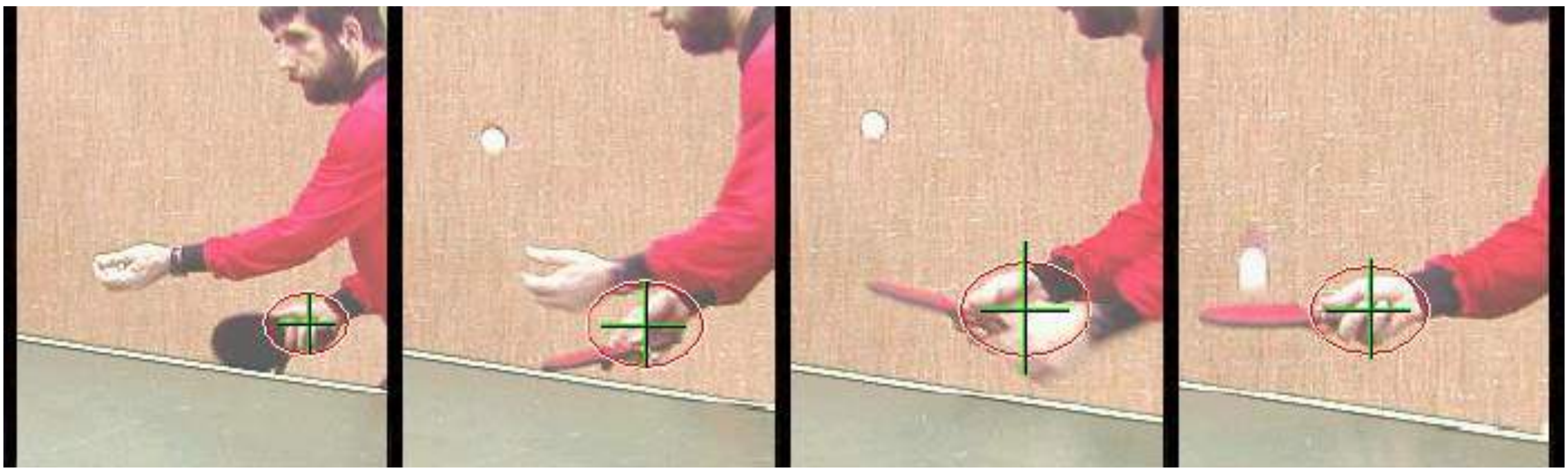
Finally… mean shift tracking in video!

Goal: find the best candidate location in frame 2



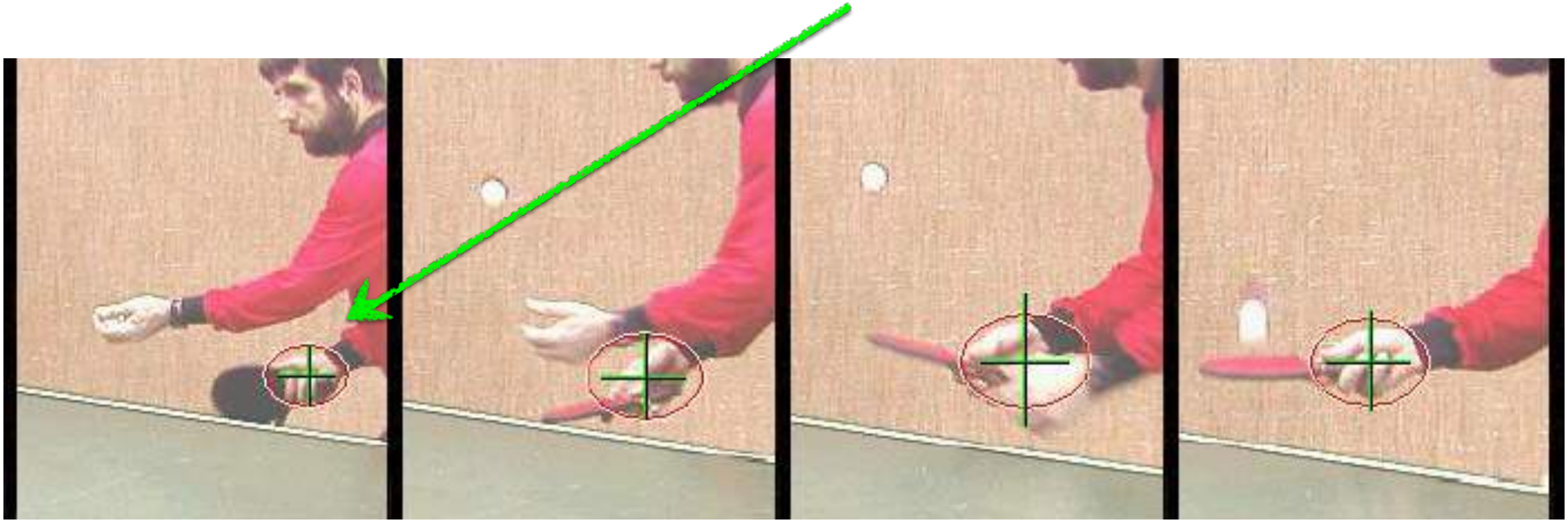
Use the mean shift algorithm
to find the best candidate location

Non-rigid object tracking



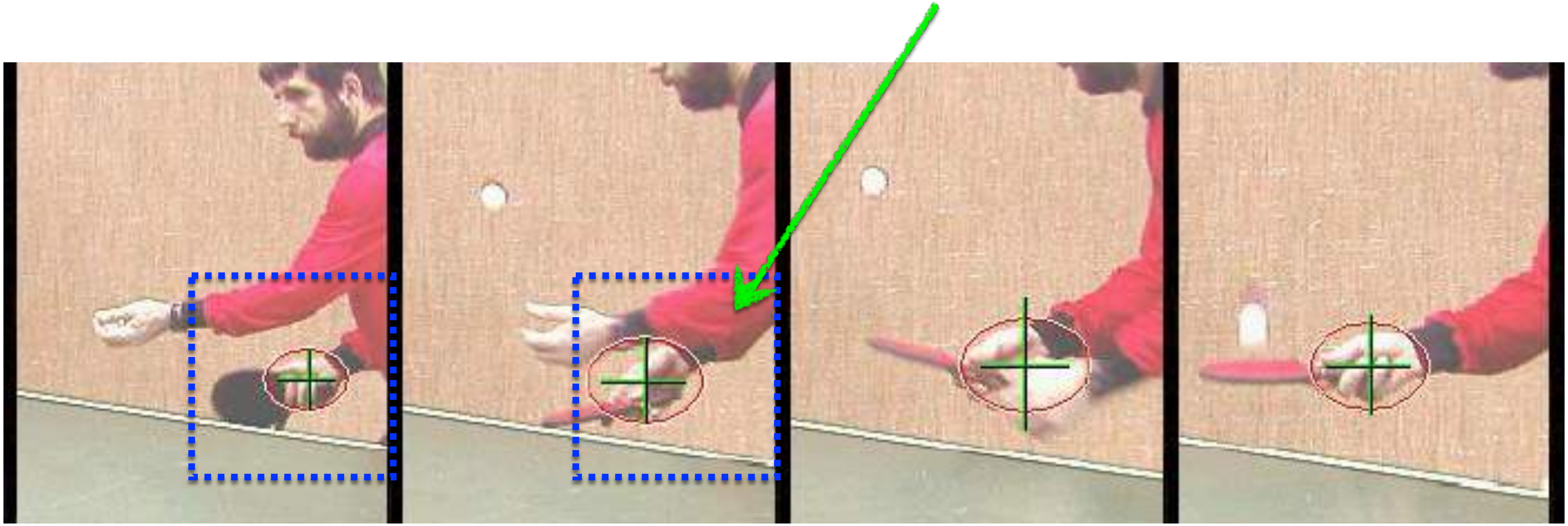
hand tracking

Compute a descriptor for the target



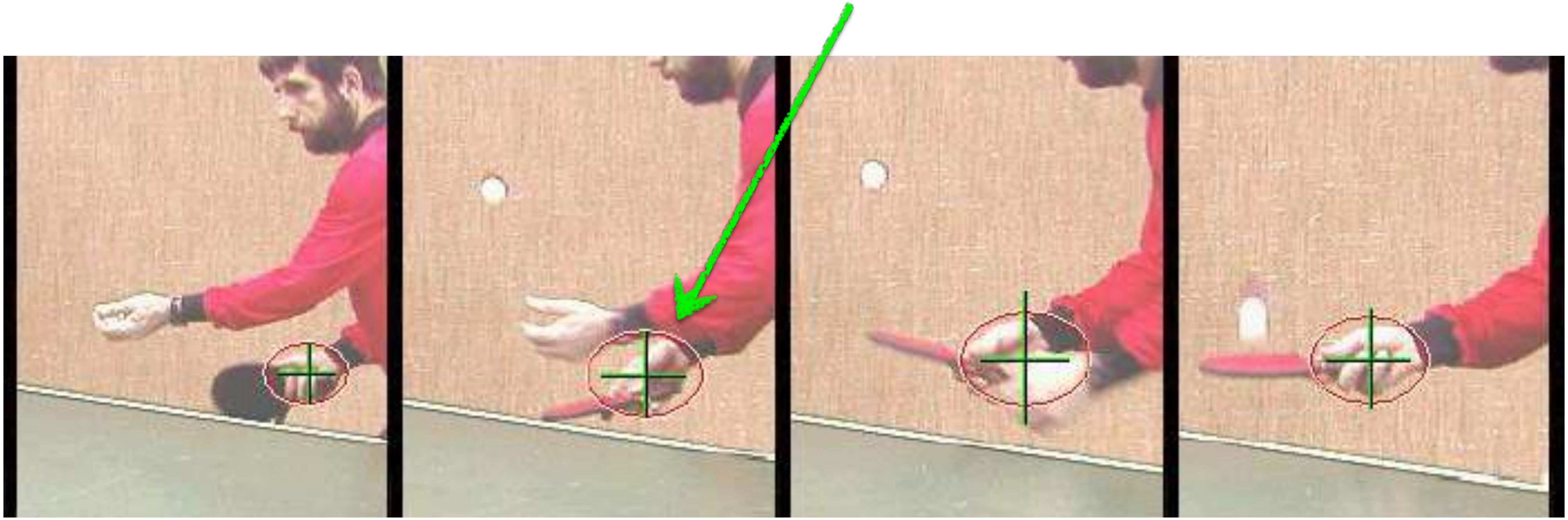
Target

Search for similar descriptor in neighborhood in next frame



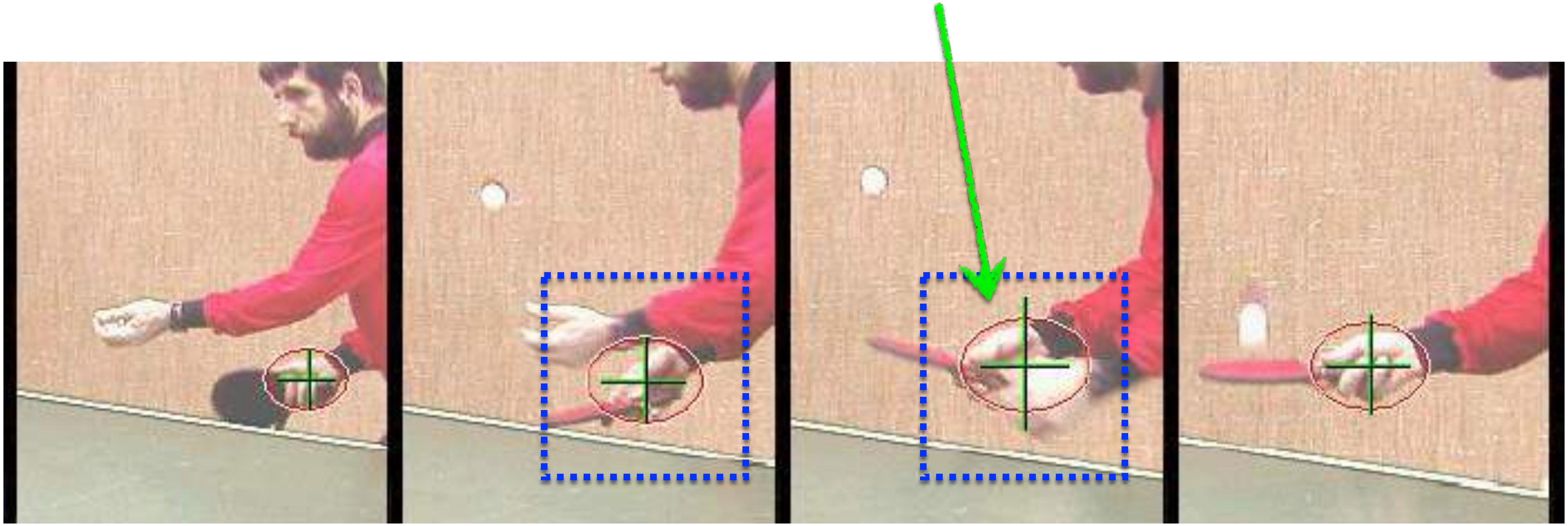
Target Candidate

Compute a descriptor for the new target



Target

Search for similar descriptor in neighborhood in next frame



Target Candidate

How do we model the target and candidate regions?

Modeling the target

M-dimensional **target** descriptor

$$\boldsymbol{q} = \{q_1, \ldots, q_M\}$$

(centered at target center)

a 'fancy' (confusing) way to write a weighted histogram

$$q_m = C \sum_n k(\|\boldsymbol{x}_n\|^2)\delta[b(\boldsymbol{x}_n) - m]$$

Normalization factor

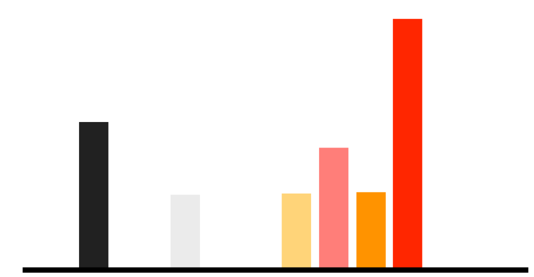
sum over all pixels

function of inverse distance (weight)

Kronecker delta function

quantization function

bin ID



A normalized color histogram (weighted by distance)

Modeling the candidate

M-dimensional **candidate** descriptor

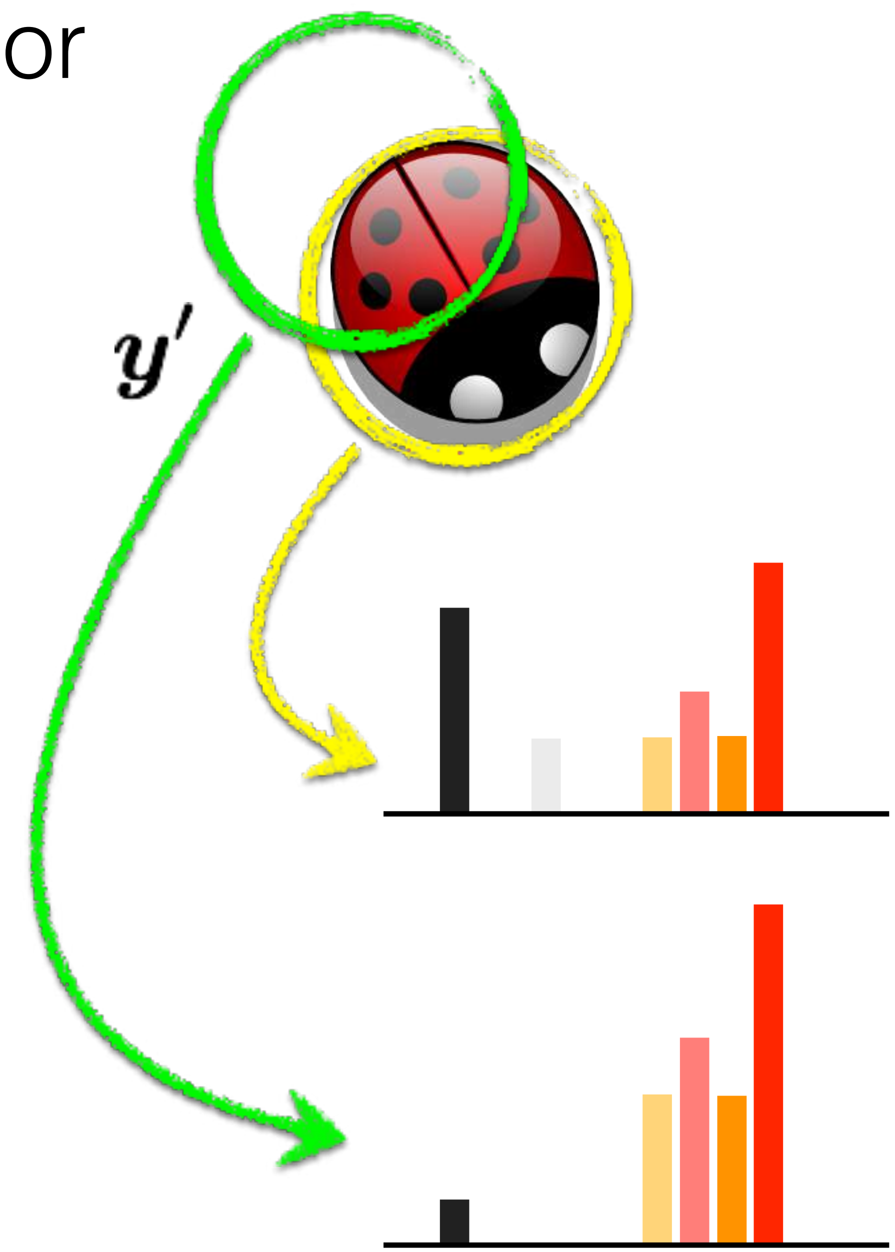
$$\boldsymbol{p}(\boldsymbol{y}) = \{p_1(\boldsymbol{y}), \ldots, p_M(\boldsymbol{y}\}$$

(centered at location **y**)

a weighted histogram at y

$$p_m = C_h \sum_n k\left(\left\|\frac{\boldsymbol{y} - \boldsymbol{x}_n}{h}\right\|^2\right) \delta[b(\boldsymbol{x}_n) - m]$$

bandwidth



Similarity between the target and candidate

Distance function

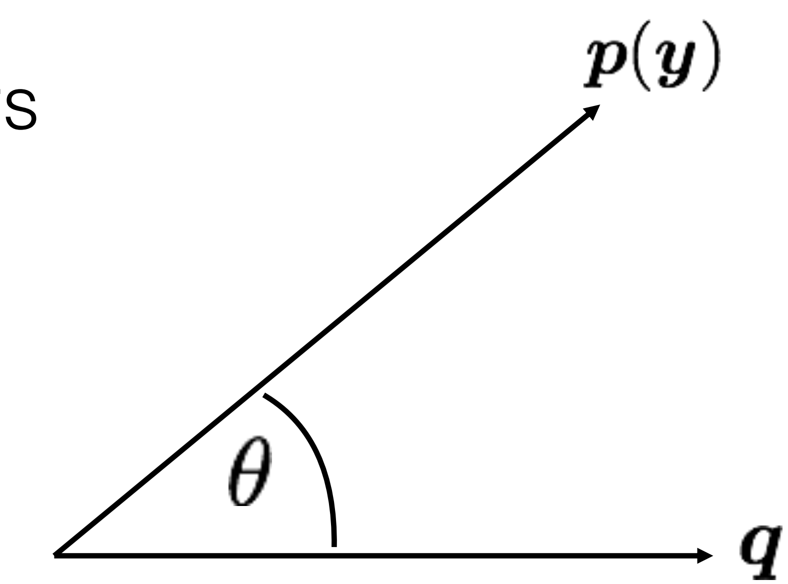
$$d(\boldsymbol{y}) = \sqrt{1 - \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]}$$

Bhattacharyya Coefficient

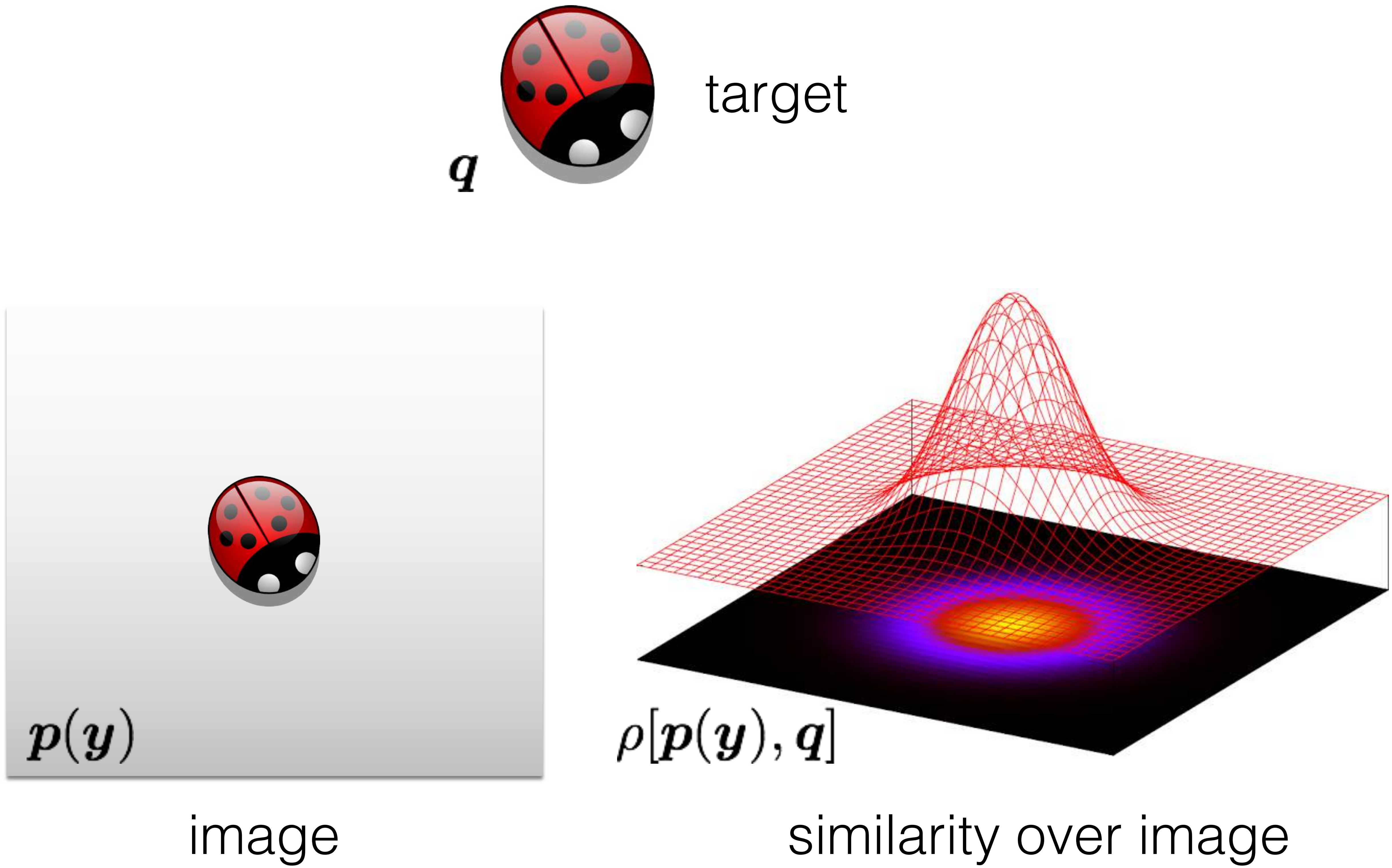
$$\rho(y) \equiv \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] = \sum_m \sqrt{p_m(\boldsymbol{y}) q_u}$$

Just the Cosine distance between two unit vectors

$$\rho(\boldsymbol{y}) = \cos\theta_{\boldsymbol{y}} = \frac{\boldsymbol{p}(\boldsymbol{y})^\top \boldsymbol{q}}{\|\boldsymbol{p}\| \|\boldsymbol{q}\|} = \sum_m \sqrt{p_m(\boldsymbol{y}) q_m}$$

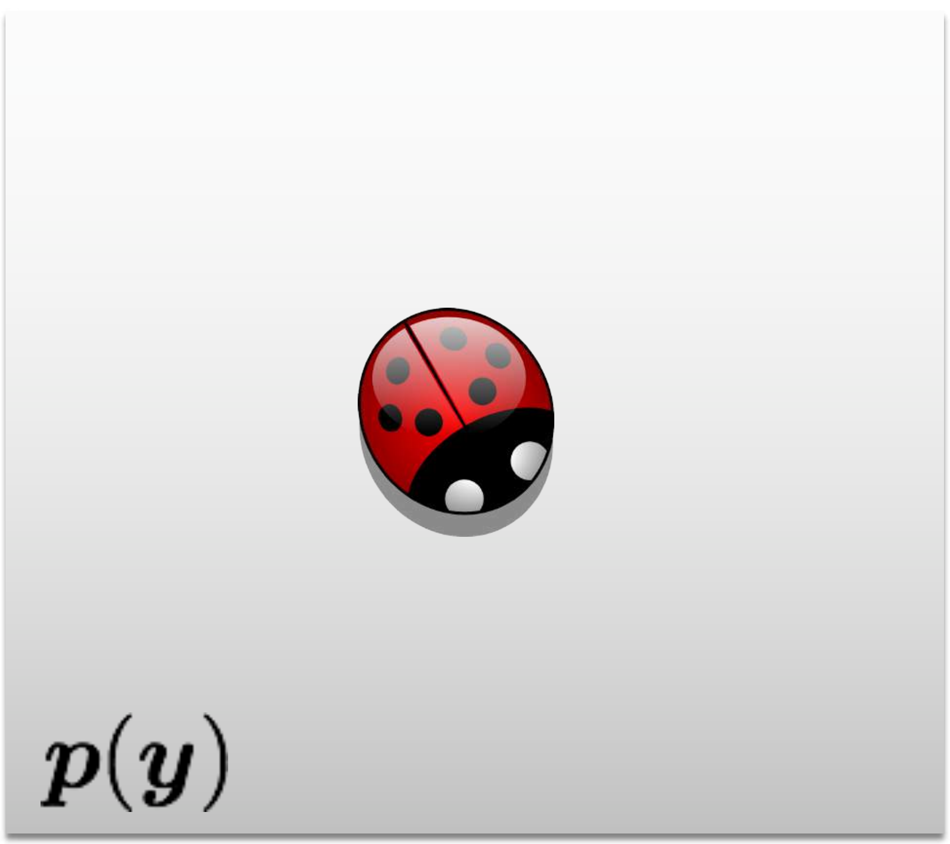


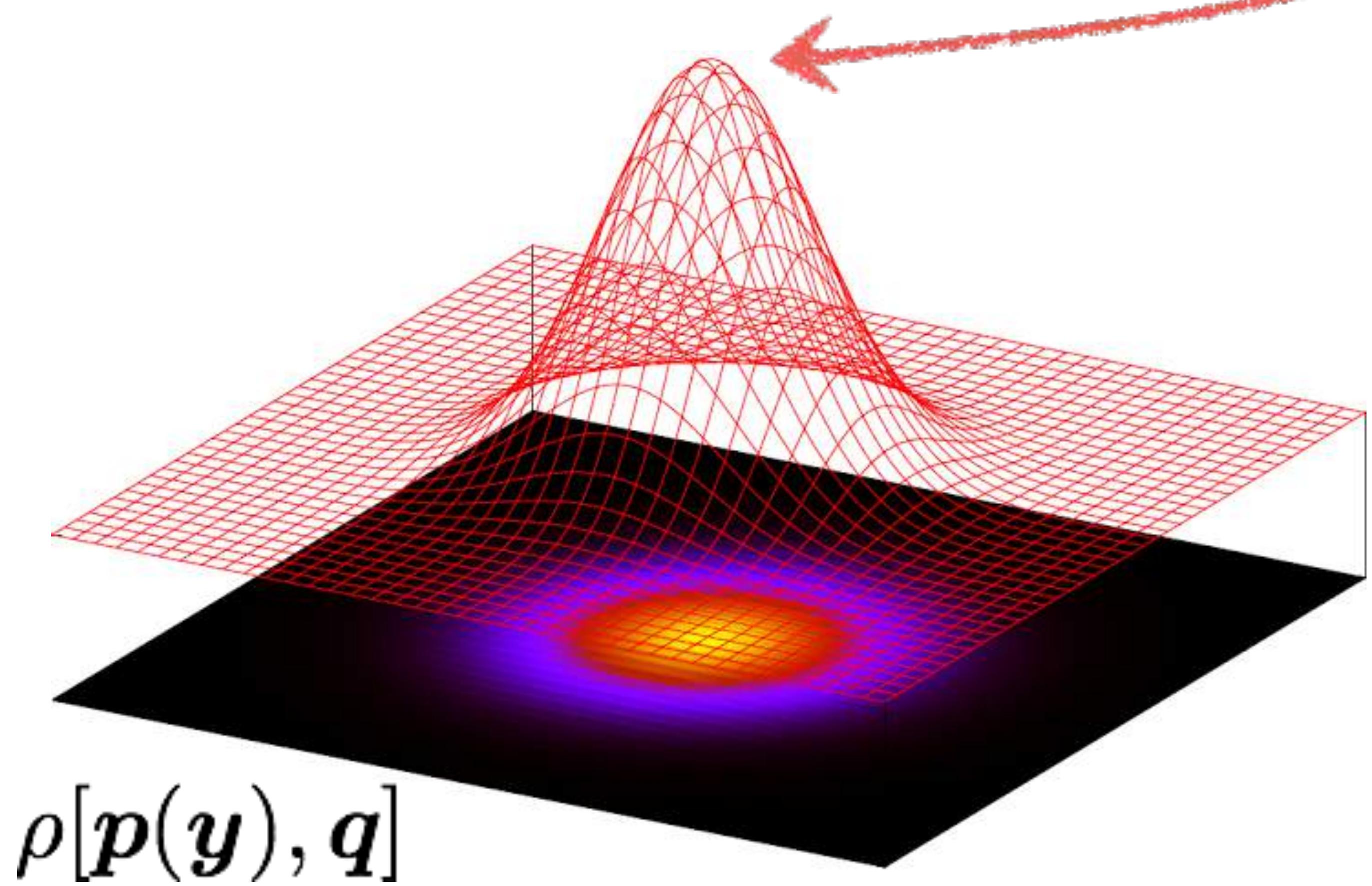
Now we can compute the similarity between
a target and multiple candidate regions


target
$\boldsymbol{q}$
$\boldsymbol{p}(\boldsymbol{y})$
$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$
image
similarity over image

target

we want to find this peak





image

similarity over image

Objective function

$$\min_{\boldsymbol{y}} d(\boldsymbol{y}) \qquad \text{same as} \qquad \max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

Assuming a good initial guess

$$\rho[\boldsymbol{p}(\boldsymbol{y}_0 + \boldsymbol{y}), \boldsymbol{q}]$$

Linearize around the initial guess (Taylor series expansion)

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \underbrace{\frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m}}_{\text{function at specified value}} + \frac{1}{2} \sum_m p_m(\boldsymbol{y}) \underbrace{\sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}}_{\text{derivative}}$$

Linearized objective

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m} + \frac{1}{2} \sum_m p_m(\boldsymbol{y}) \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}$$

$$p_m = C_h \sum_n k\left(\left\|\frac{\boldsymbol{y}-\boldsymbol{x}_n}{h}\right\|^2\right) \delta[b(\boldsymbol{x}_n) - m]$$

Remember definition of this?

Fully expanded

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m} + \frac{1}{2} \sum_m \left\{ C_h \sum_n k\left(\left\|\frac{\boldsymbol{y}-\boldsymbol{x}_n}{h}\right\|^2\right) \delta[b(\boldsymbol{x}_n) - m] \right\} \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}$$

Fully expanded linearized objective

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m} + \frac{1}{2} \sum_m \left\{ C_h \sum_n k \left(\left\| \frac{\boldsymbol{y} - \boldsymbol{x}_n}{h} \right\|^2 \right) \delta[b(\boldsymbol{x}_n) - m] \right\} \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}$$

Moving terms around…

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \boxed{\frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m}} + \boxed{\frac{C_h}{2} \sum_n w_n k \left(\left\| \frac{\boldsymbol{y} - \boldsymbol{x}_n}{h} \right\|^2 \right)}$$

Does not depend on unknown **y**

Weighted kernel density estimate

where $$w_n = \sum_m \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}} \delta[b(\boldsymbol{x}_n) - m]$$

Weight is bigger when $q_m > p_m(\boldsymbol{y}_0)$

OK, why are we doing all this math?

We want to maximize this

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

We want to maximize this

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

Fully expanded linearized objective

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m} + \frac{C_h}{2} \sum_n w_n k \left(\left\| \frac{\boldsymbol{y} - \boldsymbol{x}_n}{h} \right\|^2 \right)$$

where $\quad w_n = \sum_m \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}} \delta[b(\boldsymbol{x}_n) - m]$

We want to maximize this

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

Fully expanded linearized objective

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \cancel{\frac{1}{2} \sum_{m} \sqrt{p_m(\boldsymbol{y}_0) q_m}} + \frac{C_h}{2} \sum_{n} w_n k \left(\left\| \frac{\boldsymbol{y} - \boldsymbol{x}_n}{h} \right\|^2 \right)$$

doesn't depend on unknown **y**

where $$w_n = \sum_m \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}} \delta[b(\boldsymbol{x}_n) - m]$$

We want to maximize this

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

only need to maximize this!

Fully expanded linearized objective

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \cancel{\frac{1}{2}\sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m}} + \frac{C_h}{2}\sum_n w_n k\left(\left\|\frac{\boldsymbol{y}-\boldsymbol{x}_n}{h}\right\|^2\right)$$

doesn't depend on unknown **y**

where $\quad w_n = \sum_m \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}\delta[b(\boldsymbol{x}_n) - m]$

We want to maximize this

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

Fully expanded linearized objective

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \cancel{\frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m}} + \frac{C_h}{2} \sum_n w_n k \left(\left\| \frac{\boldsymbol{y} - \boldsymbol{x}_n}{h} \right\|^2 \right)$$

doesn't depend on unknown **y**

where $$w_n = \sum_m \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}} \delta[b(\boldsymbol{x}_n) - m]$$

what can we use to solve this weighted KDE?

Mean Shift Algorithm!

$$\frac{C_h}{2} \sum_n w_n k \left(\left\| \frac{\boldsymbol{y} - \boldsymbol{x}_n}{h} \right\|^2 \right)$$

the new sample of mean of this KDE is

$$\boldsymbol{y}_1 = \frac{\sum_n \boldsymbol{x}_n w_n g \left(\left\| \frac{\boldsymbol{y}_0 - \boldsymbol{x}_n}{h} \right\|^2 \right)}{\sum_n w_n g \left(\left\| \frac{\boldsymbol{y}_0 - \boldsymbol{x}_n}{h} \right\|^2 \right)}$$

(new candidate location)

(this was derived earlier)

Mean-Shift Object Tracking

For each frame:

1. Initialize location $\boldsymbol{y}_0$
   Compute $\boldsymbol{q}$
   Compute $\boldsymbol{p}(\boldsymbol{y}_0)$

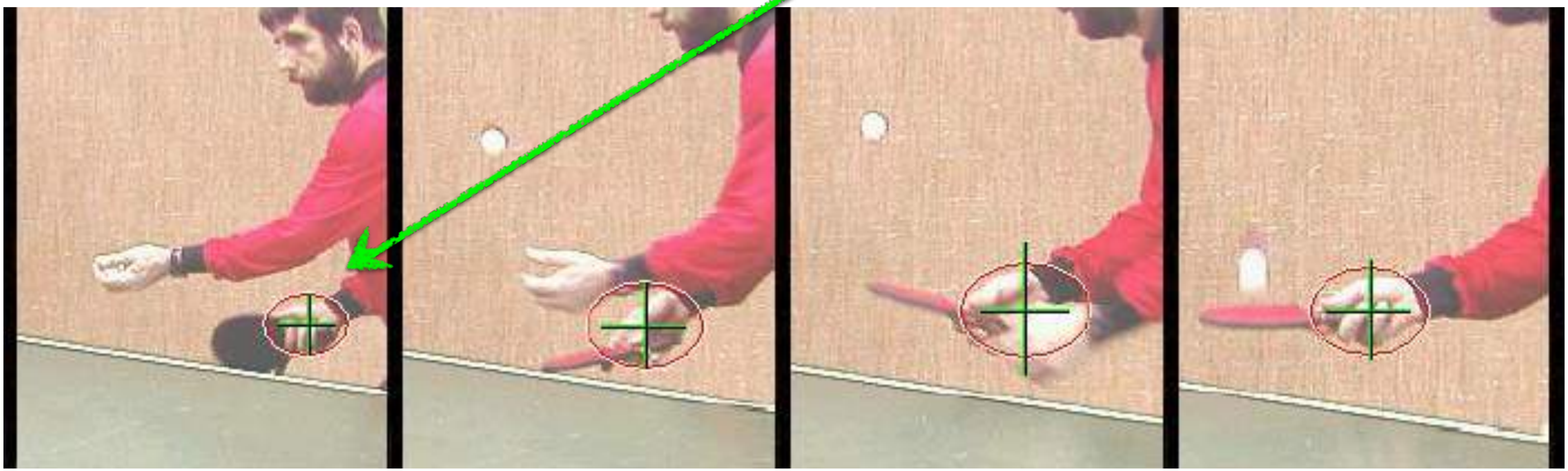
2. Derive weights w_n

3. Shift to new candidate location (mean shift) $\boldsymbol{y}_1$

4. Compute $\boldsymbol{p}(\boldsymbol{y}_1)$

5. If $\|\boldsymbol{y}_0 - \boldsymbol{y}_1\| < \epsilon$ return
   Otherwise $\boldsymbol{y}_0 \leftarrow \boldsymbol{y}_1$ and go back to 2

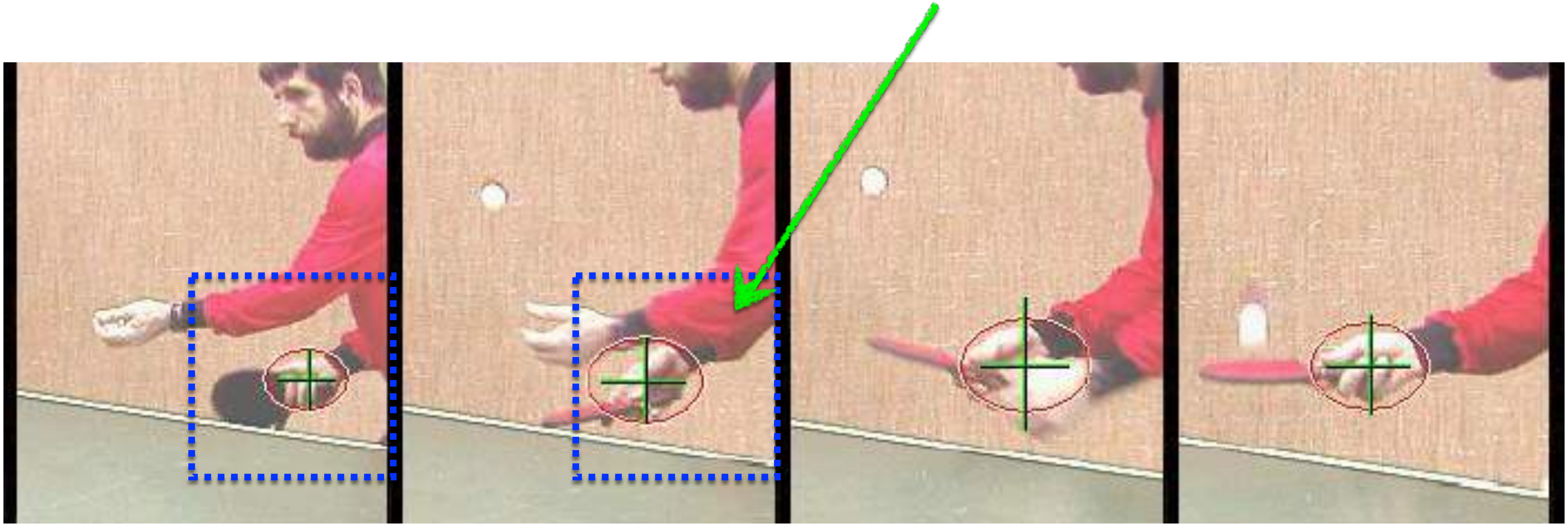
Compute a descriptor for the target



Target

$\boldsymbol{q}$

Search for similar descriptor in neighborhood in next frame

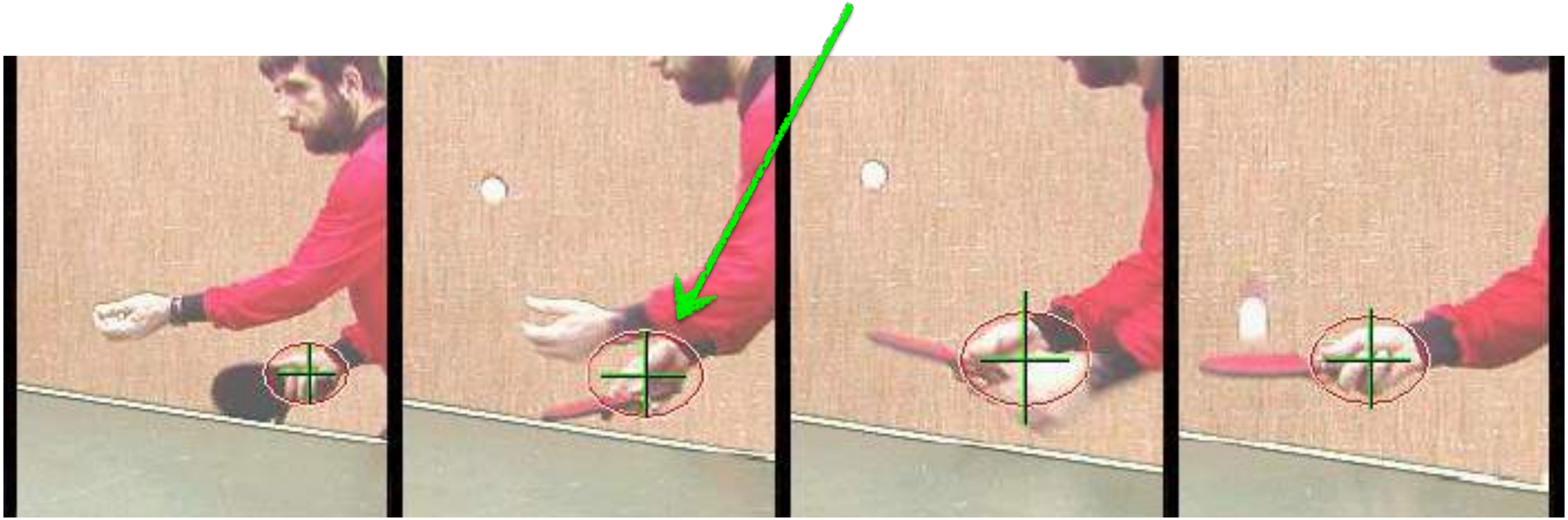


Target

Candidate

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

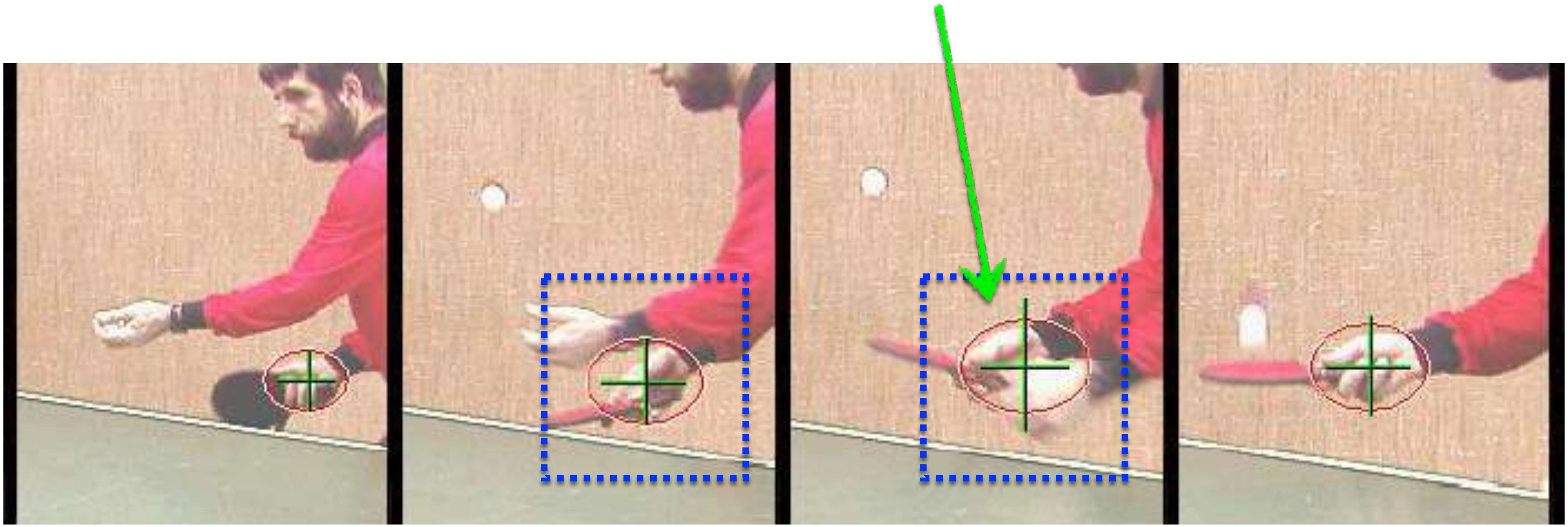
Compute a descriptor for the new target



Target

$\boldsymbol{q}$

Search for similar descriptor in neighborhood in next frame



Target Candidate

$$\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$$

Modern trackers

Learning Multi-Domain Convolutional Neural Networks for Visual Tracking

Hyeonseob Nam and Bohyung Han

References

Basic reading:

- Szeliski, Sections 4.1.4, 5.3, 8.1.