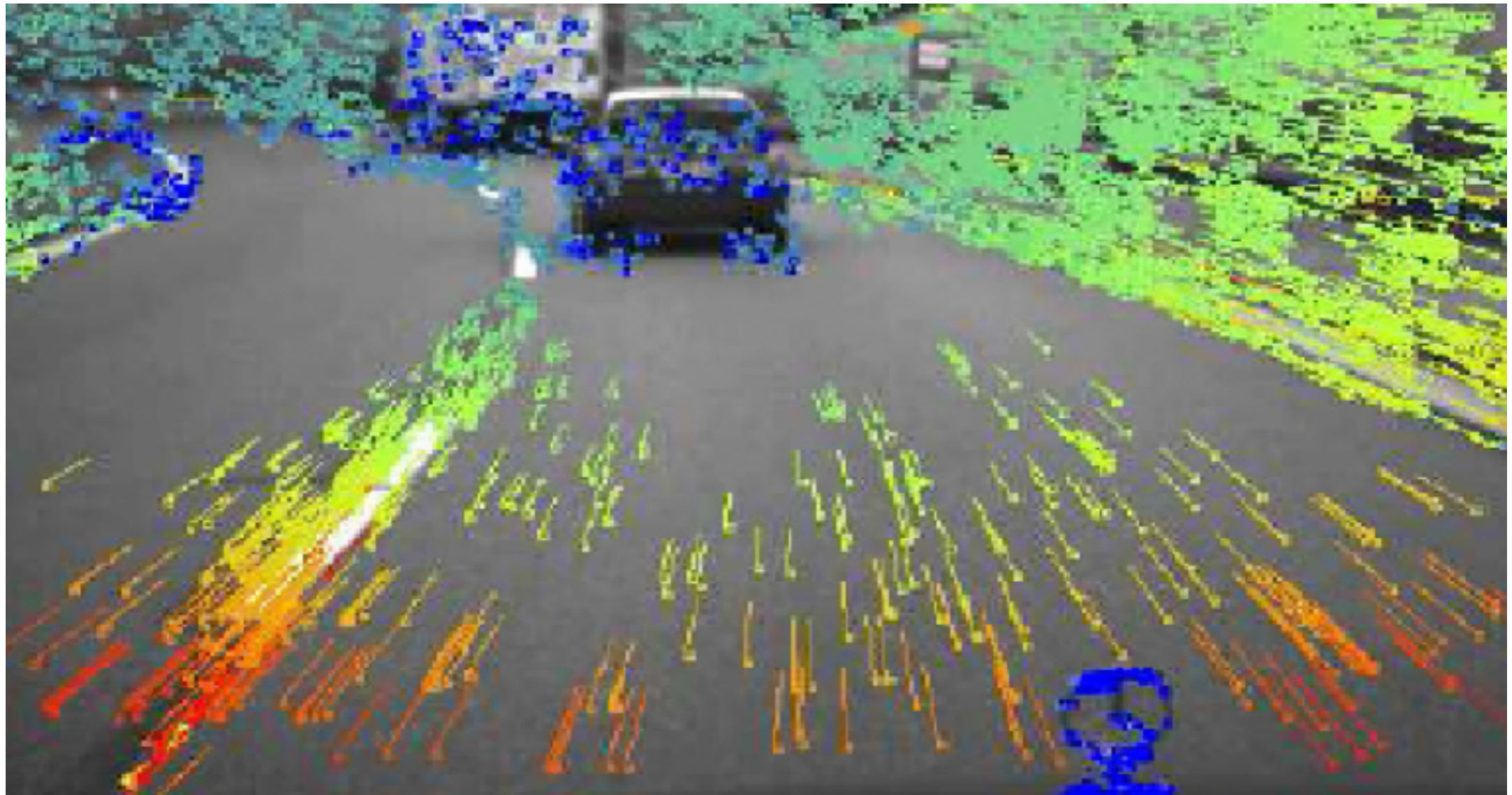


# Optical flow



# Overview of today's lecture

- Quick intro to vision for video.
- Optical flow.
- Constant flow.
- Horn-Schunck flow.

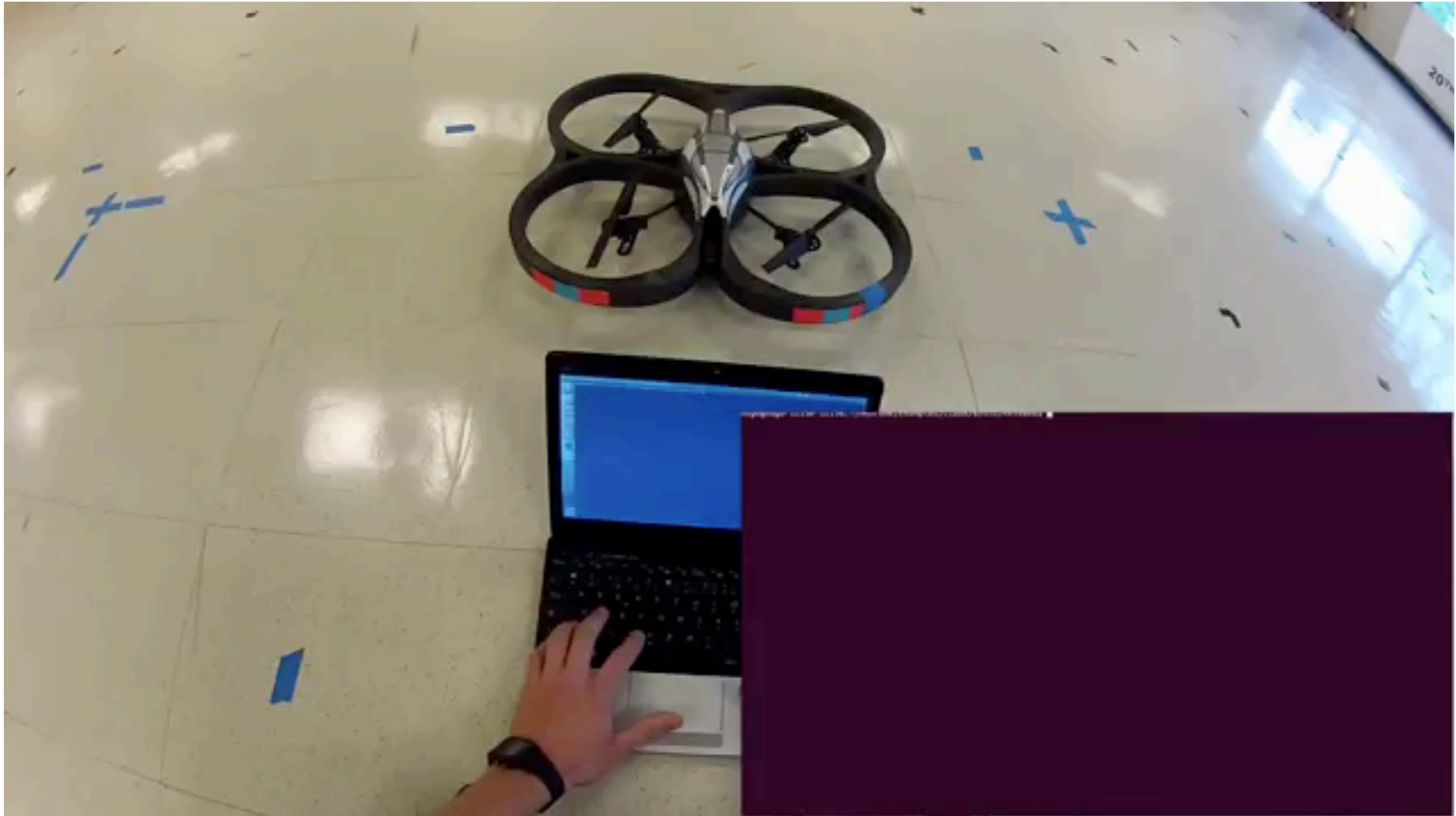
# Slide credits

Most of these slides were adapted from:

- Kris Kitani (16-385, Spring 2017).

# Computer vision for video

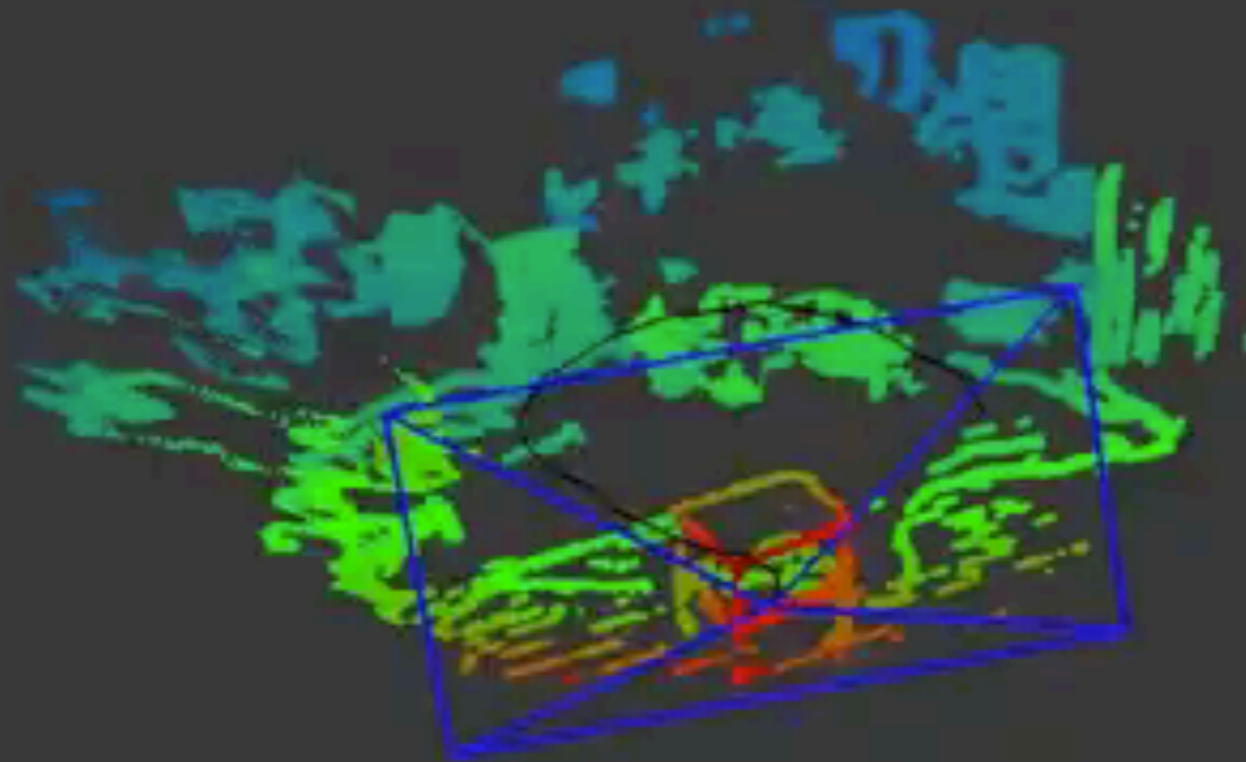
# Optical flow used for feature tracking on a drone



# optical flow used for motion estimation in visual odometry

00:00:08.000

camera image



It was captured in a motion capture system,  
which is reason for the flickering lights.

# Optical flow

# Optical Flow

## **Problem Definition**

Given two consecutive image frames,  
estimate the motion of each pixel

## **Assumptions**

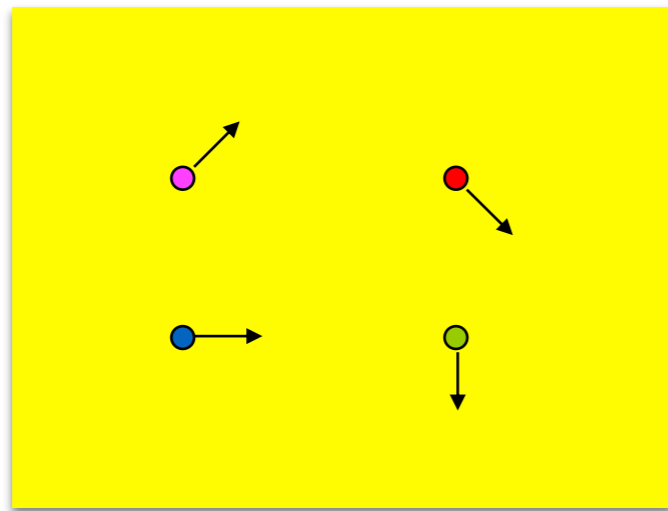
Brightness constancy

Small motion

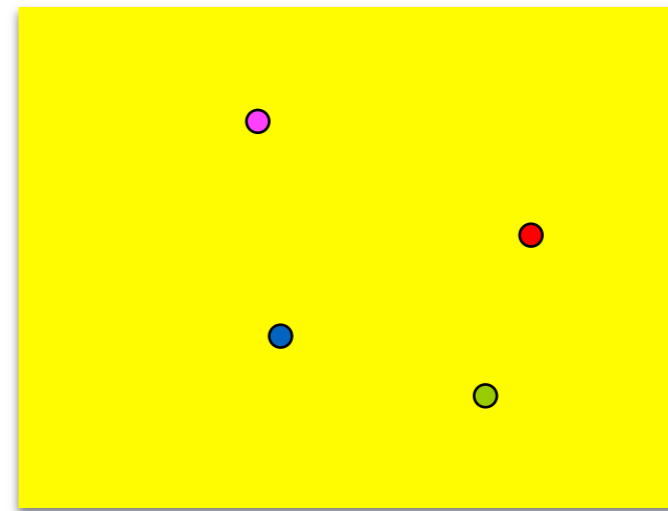


# Optical Flow

(Problem definition)



$I(x, y, t)$



$I(x, y, t')$



Estimate the motion  
(flow) between these  
two consecutive images



*How is this different from estimating a 2D transform?*

# Key Assumptions

(unique to optical flow)

## Color Constancy

(Brightness constancy for intensity images)

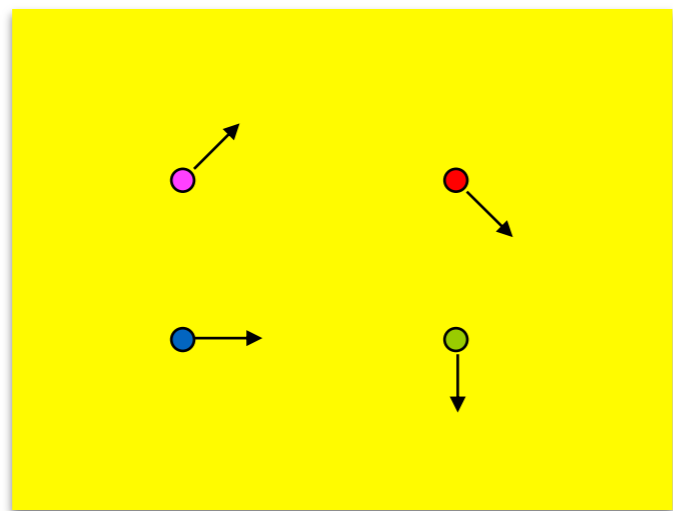
Implication: allows for pixel to pixel comparison  
(not image features)

## Small Motion

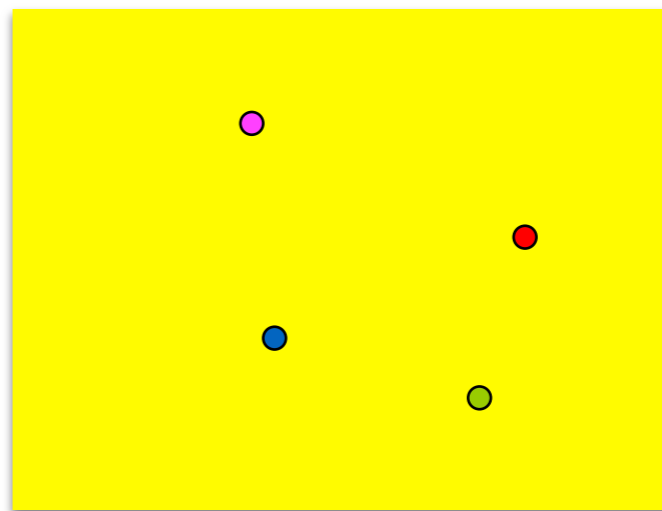
(pixels only move a little bit)

Implication: linearization of the brightness  
constancy constraint

# Approach



$I(x, y, t)$



$I(x, y, t')$

Look for nearby pixels with the same color

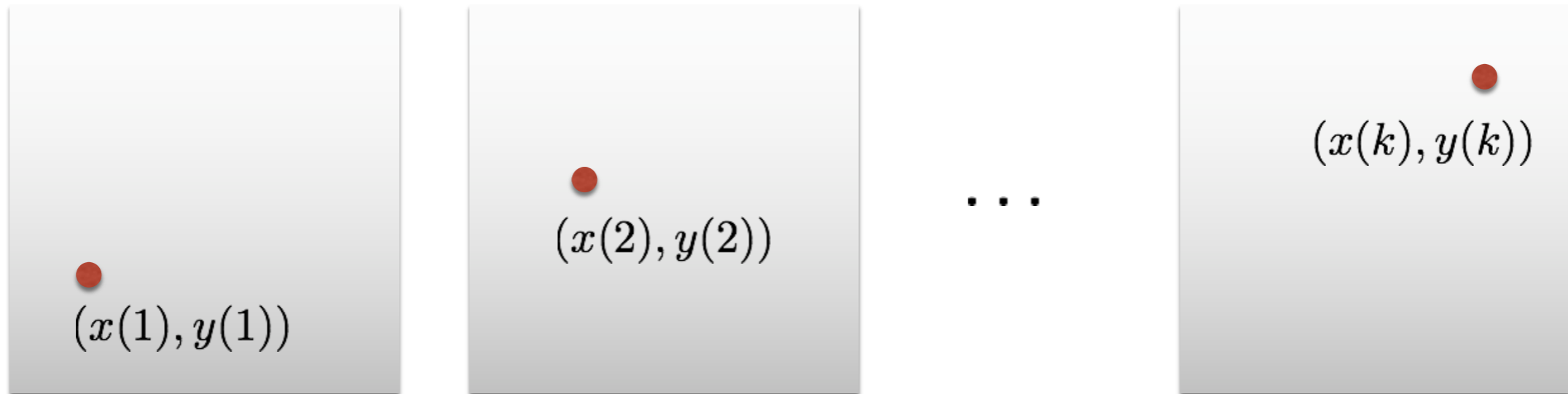
(small motion)

(color constancy)

Assumption 1

# Brightness constancy

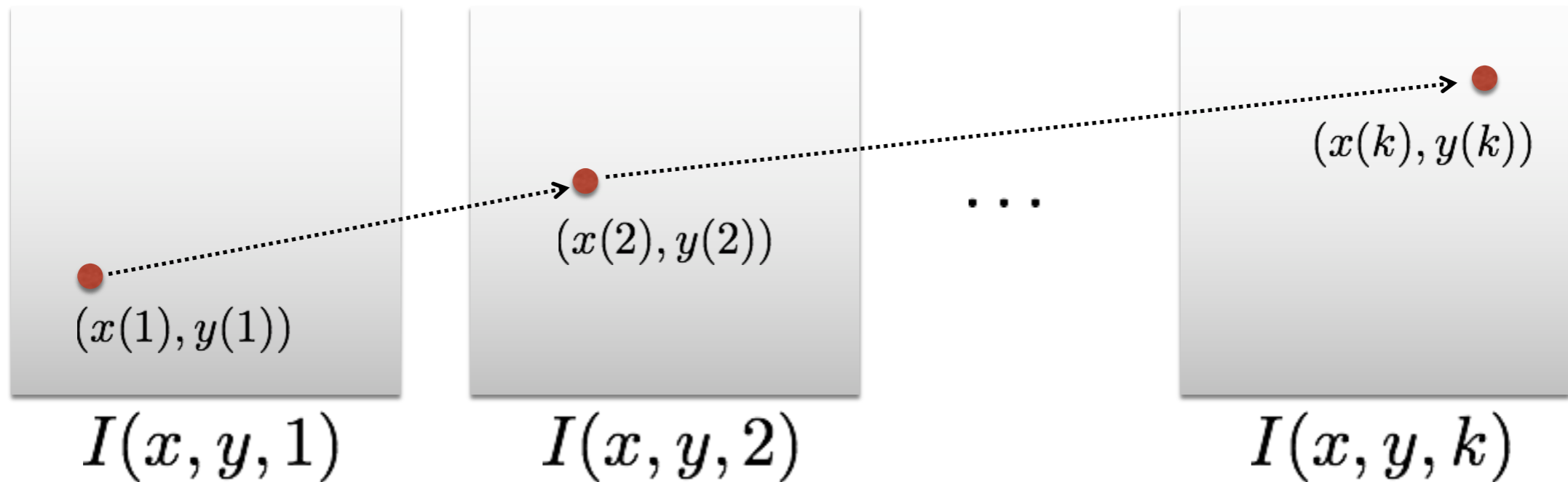
Scene point moving through image sequence



Assumption 1

# Brightness constancy

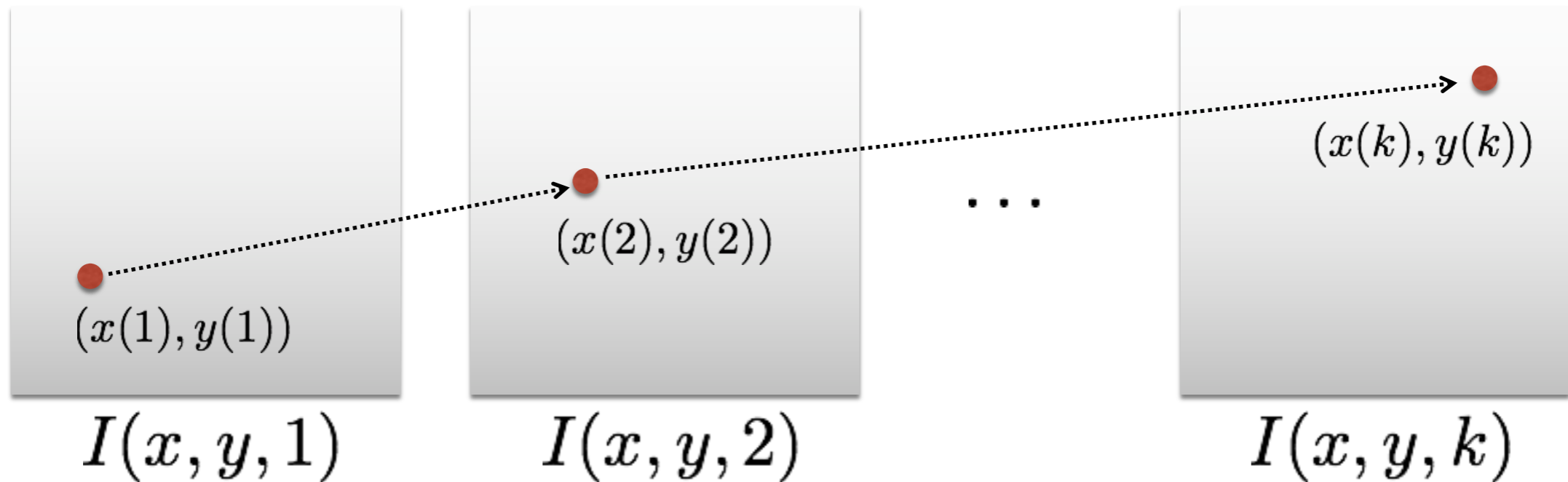
Scene point moving through image sequence



Assumption 1

# Brightness constancy

Scene point moving through image sequence

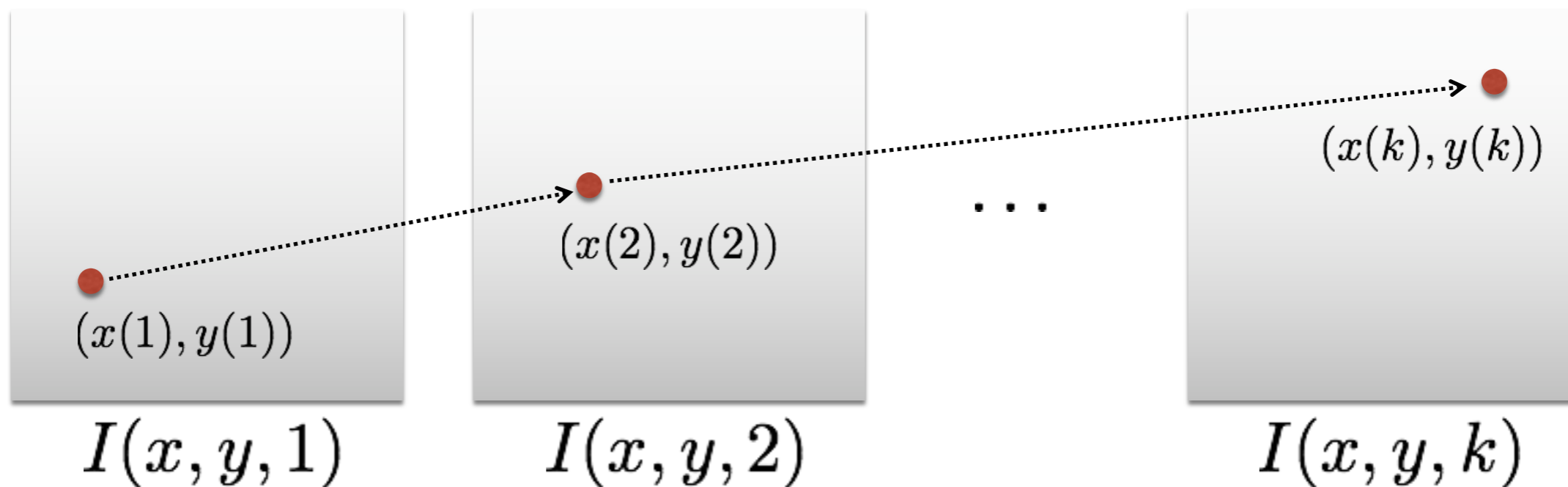


**Assumption: Brightness of the point will remain the same**

## Assumption 1

# Brightness constancy

Scene point moving through image sequence



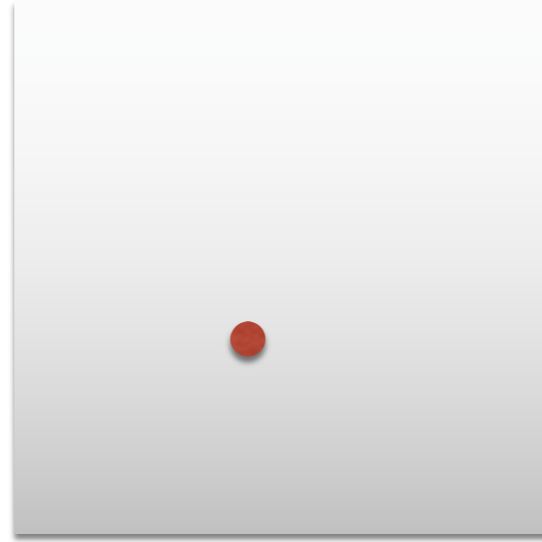
**Assumption: Brightness of the point will remain the same**

$$I(x(t), y(t), t) = C$$

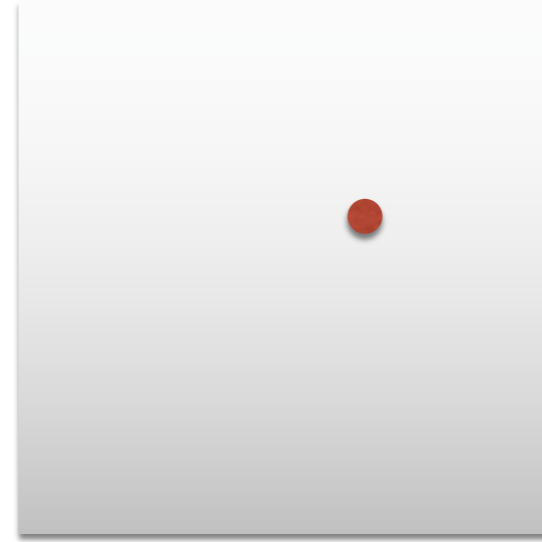
constant

Assumption 2

# Small motion



$I(x, y, t)$

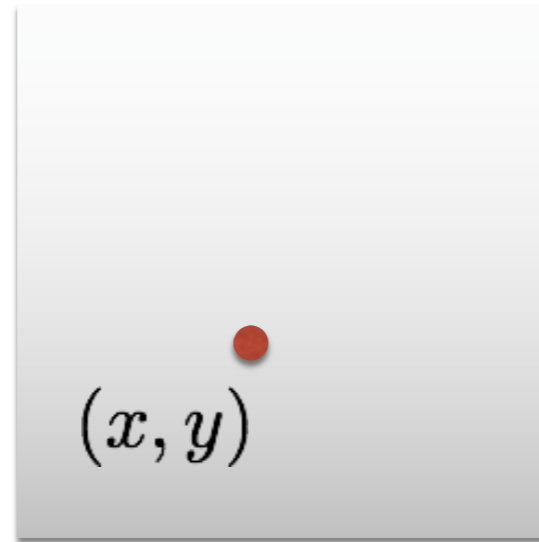


$I(x, y, t + \delta t)$

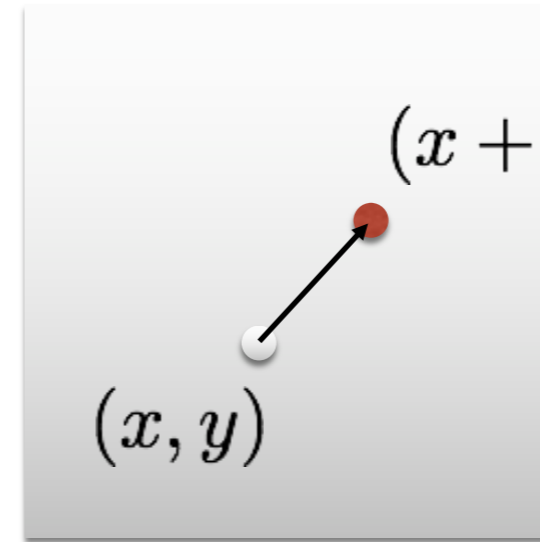


## Assumption 2

# Small motion



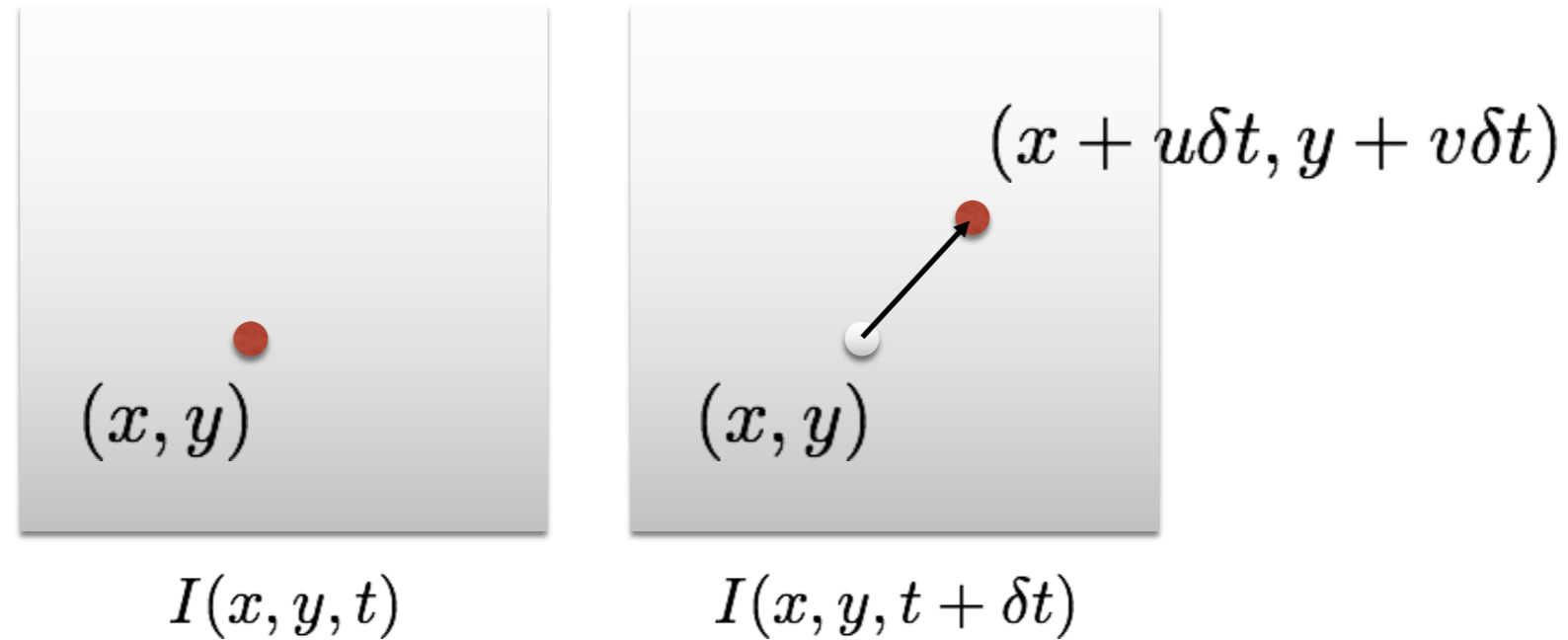
$I(x, y, t)$



$I(x, y, t + \delta t)$

## Assumption 2

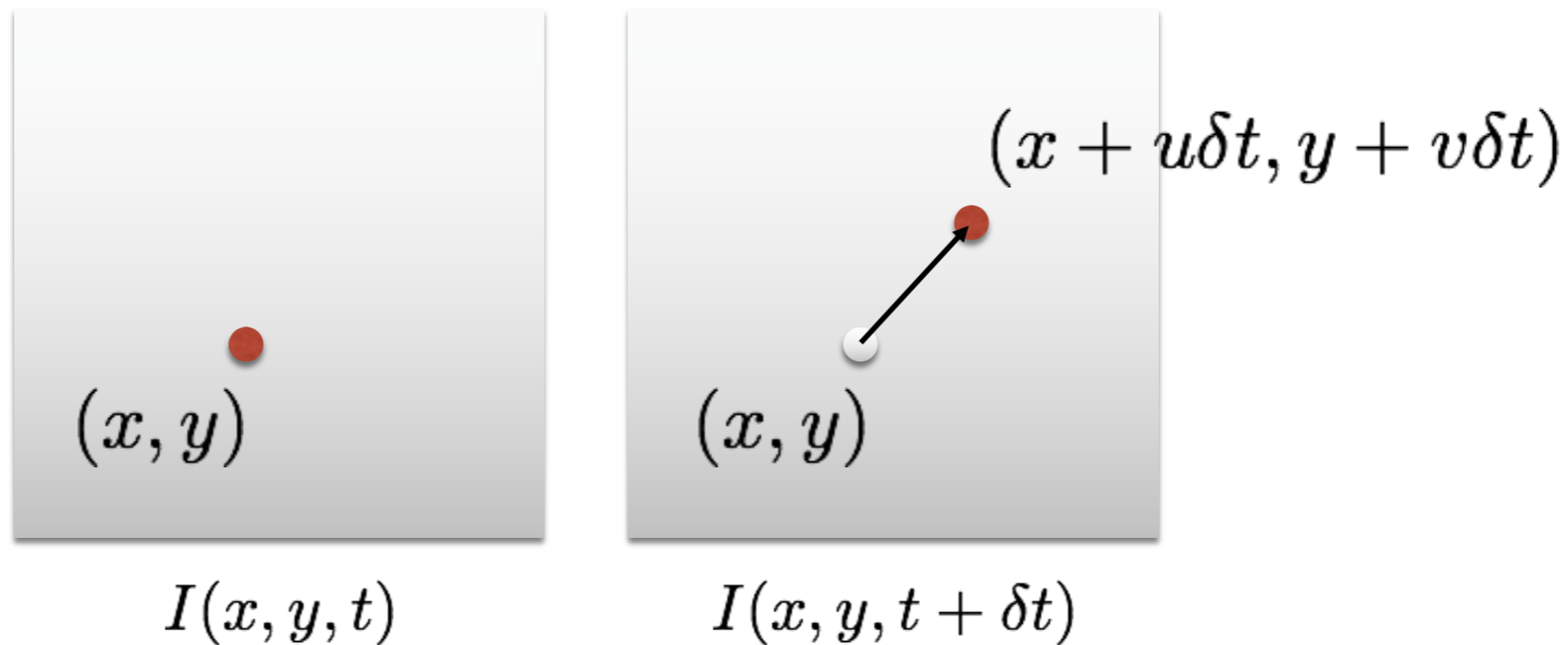
# Small motion



Optical flow (velocities):  $(u, v)$       Displacement:  $(\delta x, \delta y) = (u\delta t, v\delta t)$

## Assumption 2

# Small motion



Optical flow (velocities):  $(u, v)$       Displacement:  $(\delta x, \delta y) = (u\delta t, v\delta t)$

For a really small space-time step...

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

... the brightness between two consecutive image frames is the same

These assumptions yield the ...

## Brightness Constancy Equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative

partial derivative

*Equation is not obvious. Where does this come from?*

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

**Insight:**

If the time step is really small,  
we can *linearize* the intensity function

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t) \quad \text{assuming small motion}$$



$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

partial derivative

$$I(x, y, t) + \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = I(x, y, t) \quad \text{assuming small motion}$$

fixed point

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0 \quad \text{cancel terms}$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0 \quad \begin{array}{l} \text{divide by } \delta t \\ \text{take limit } \delta t \rightarrow 0 \end{array}$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$\frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = 0 \quad \begin{array}{l} \text{divide by } \delta t \\ \text{take limit } \delta t \rightarrow 0 \end{array}$$

$$\frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

## Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0 \quad \begin{array}{l} \text{divide by } \delta t \\ \text{take limit } \delta t \rightarrow 0 \end{array}$$

$$\frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

**Brightness Constancy Equation**

$$\frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

**Brightness  
Constancy Equation**

$$I_x u + I_y v + I_t = 0$$

(x-flow)          (y-flow)

shorthand notation

$$\nabla I^\top \mathbf{v} + I_t = 0$$

(1 x 2)      (2 x 1)

vector form

(putting the math aside for a second...)

What do the terms of the  
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

(putting the math aside for a second...)

What do the terms of the  
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

Image gradients  
(at a point p)





(putting the math aside for a second...)

What do the terms of the  
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients  
(at a point p)

(putting the math aside for a second...)

What do the terms of the  
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients  
(at a point p)

temporal gradient

*How do you compute these terms?*

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference

Sobel filter

Derivative-of-Gaussian filter

...

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference

Sobel filter

Derivative-of-Gaussian filter

...

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference

Sobel filter

Derivative-of-Gaussian filter

...

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

frame differencing

# Frame differencing

$$I_t = \frac{\partial I}{\partial t}$$

	$t$			
1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

-

	$t + 1$			
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

=

0	0	0	0	0
0	0	0	0	0
0	9	9	9	9
0	9	0	0	0
0	9	0	0	0
0	9	0	0	0

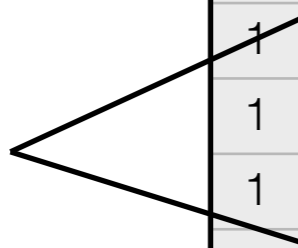
(example of a forward difference)

Example:

$t$

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

3 x 3 patch



$t + 1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

$$I_x = \frac{\partial I}{\partial x}$$

-	0	0	0	-
-	0	0	0	-
-	9	0	0	-
-	9	0	0	-
-	9	0	0	-
-	9	0	0	-

$$I_y = \frac{\partial I}{\partial y}$$

-	-	-	-	-
0	0	0	0	0
0	9	9	9	9
-	0	0	0	0
0	0	0	0	0
-	-	-	-	-

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	9	9	9	9
0	9	0	0	0
0	9	0	0	0
0	9	0	0	0

-1 0 1

-1  
0  
1



$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference

Sobel filter

Derivative-of-Gaussian filter

...

How do you compute this?

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference

Sobel filter

Derivative-of-Gaussian filter

...

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

**We need to solve for this!**

(this is the unknown in the  
optical flow problem)

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference

Sobel filter

Derivative-of-Gaussian filter

...

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$(u, v)$

Solution lies on a line

Cannot be found uniquely  
with a single constraint

$$I_t = \frac{\partial I}{\partial t}$$

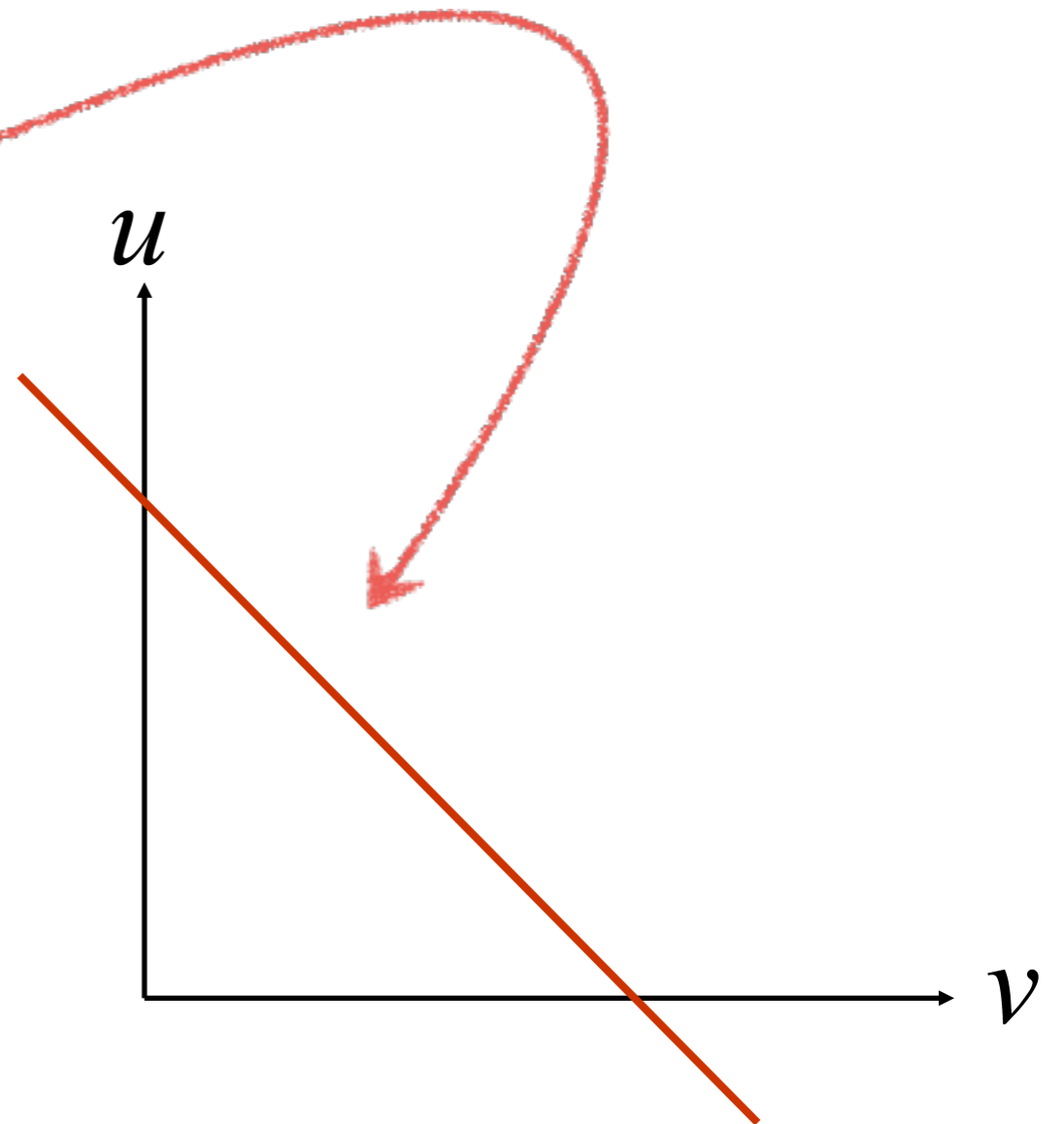
**temporal derivative**

frame differencing

Solution lies on a straight line

$$I_x u + I_y v + I_t = 0$$

many combinations of  $u$  and  $v$  will satisfy the equality



The solution cannot be determined uniquely with a single constraint (a single pixel)

**unknown**

$$I_x u + I_y v + I_t = 0$$

**known**

*We need at least \_\_\_\_\_ equations to solve for 2 unknowns.*

**unknown**

$$I_x u + I_y v + I_t = 0$$

**known**

*Where do we get more equations (constraints)?*

Constant flow

*Where do we get more equations (constraints)?*

$$I_x u + I_y v + I_t = 0$$

Assume that the surrounding patch (say 5x5) has  
**'constant flow'**



# Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a  $5 \times 5$  image patch, gives us  equations

## Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$I_x(\mathbf{p}_1)u + I_y(\mathbf{p}_1)v = -I_t(\mathbf{p}_1)$$

$$I_x(\mathbf{p}_2)u + I_y(\mathbf{p}_2)v = -I_t(\mathbf{p}_2)$$

⋮

$$I_x(\mathbf{p}_{25})u + I_y(\mathbf{p}_{25})v = -I_t(\mathbf{p}_{25})$$

Equivalent to solving:

$$A^T A \hat{x} = A^T b$$
$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum_{p \in P} I_x I_t \\ \sum_{p \in P} I_y I_t \end{bmatrix}$$

where the summation is over each pixel  $p$  in patch  $P$

$$x = (A^T A)^{-1} A^T b$$

Equivalent to solving:

$$A^T A \hat{x} = A^T b$$
$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum_{p \in P} I_x I_t \\ \sum_{p \in P} I_y I_t \end{bmatrix}$$

where the summation is over each pixel  $p$  in patch  $P$

Sometimes called 'Lucas-Kanade Optical Flow'

(can be interpreted to be a special case of the LK method with a translational warp model)

When is this solvable?

$$A^{\top} A \hat{x} = A^{\top} b$$

When is this solvable?

$$A^T A \hat{x} = A^T b$$

$A^T A$  should be invertible

$A^T A$  should not be too small

$\lambda_1$  and  $\lambda_2$  should not be too small

$A^T A$  should be well conditioned

$\lambda_1/\lambda_2$  should not be too large ( $\lambda_1$ =larger eigenvalue)

*Where have you seen this before?*

$$A^{\top} A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

*Where have you seen this before?*

$$A^T A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!



*Where have you seen this before?*

$$A^T A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!

*What are the implications?*

# Implications

- Corners are when  $\lambda_1, \lambda_2$  are big; this is also when Lucas-Kanade optical flow works best
- Corners are regions with two different directions of gradient (at least)
- Corners are good places to compute flow!

*What happens when you have no 'corners'?*

*You want to compute optical flow.  
What happens if the image patch contains only a line?*

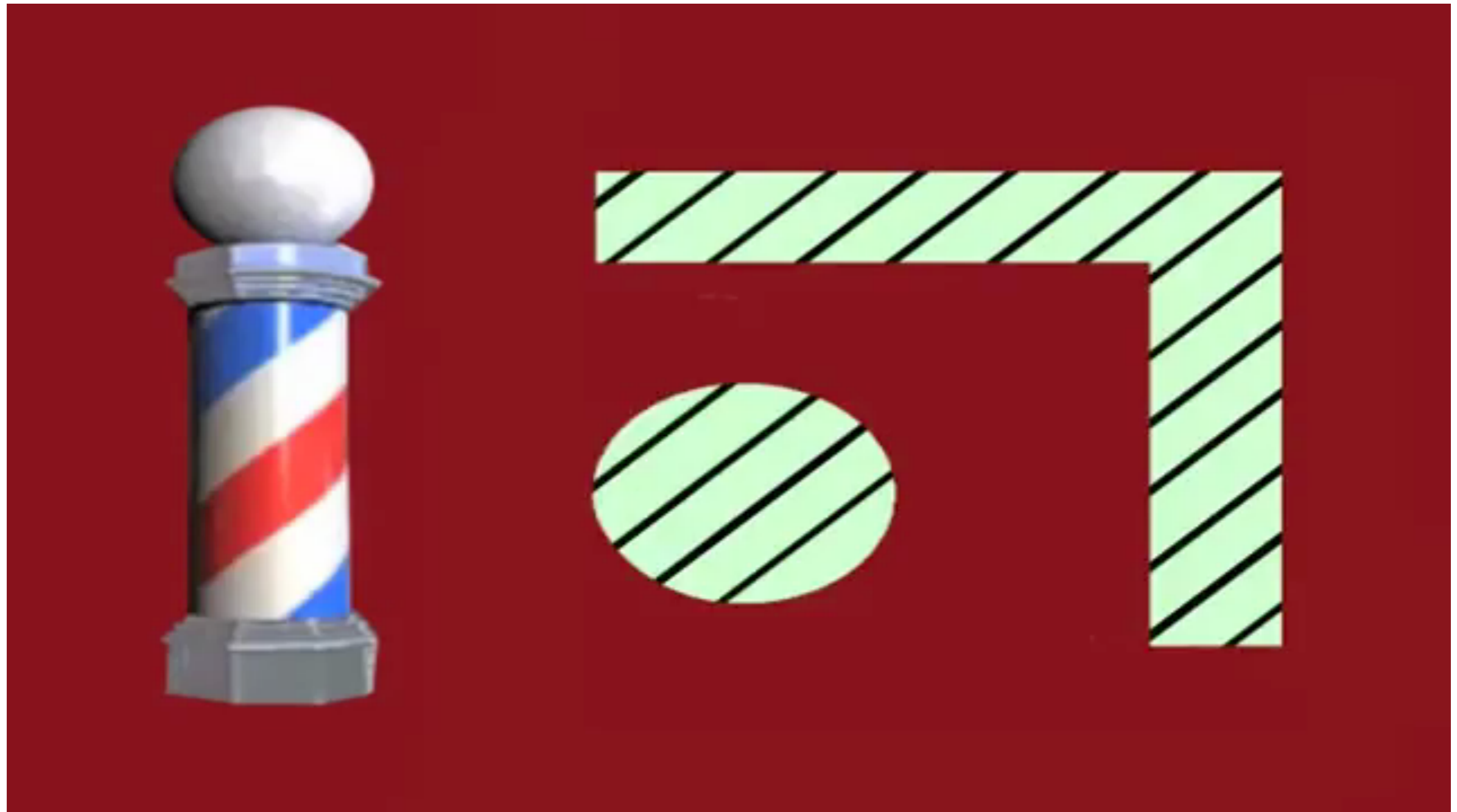
# Barber's pole illusion



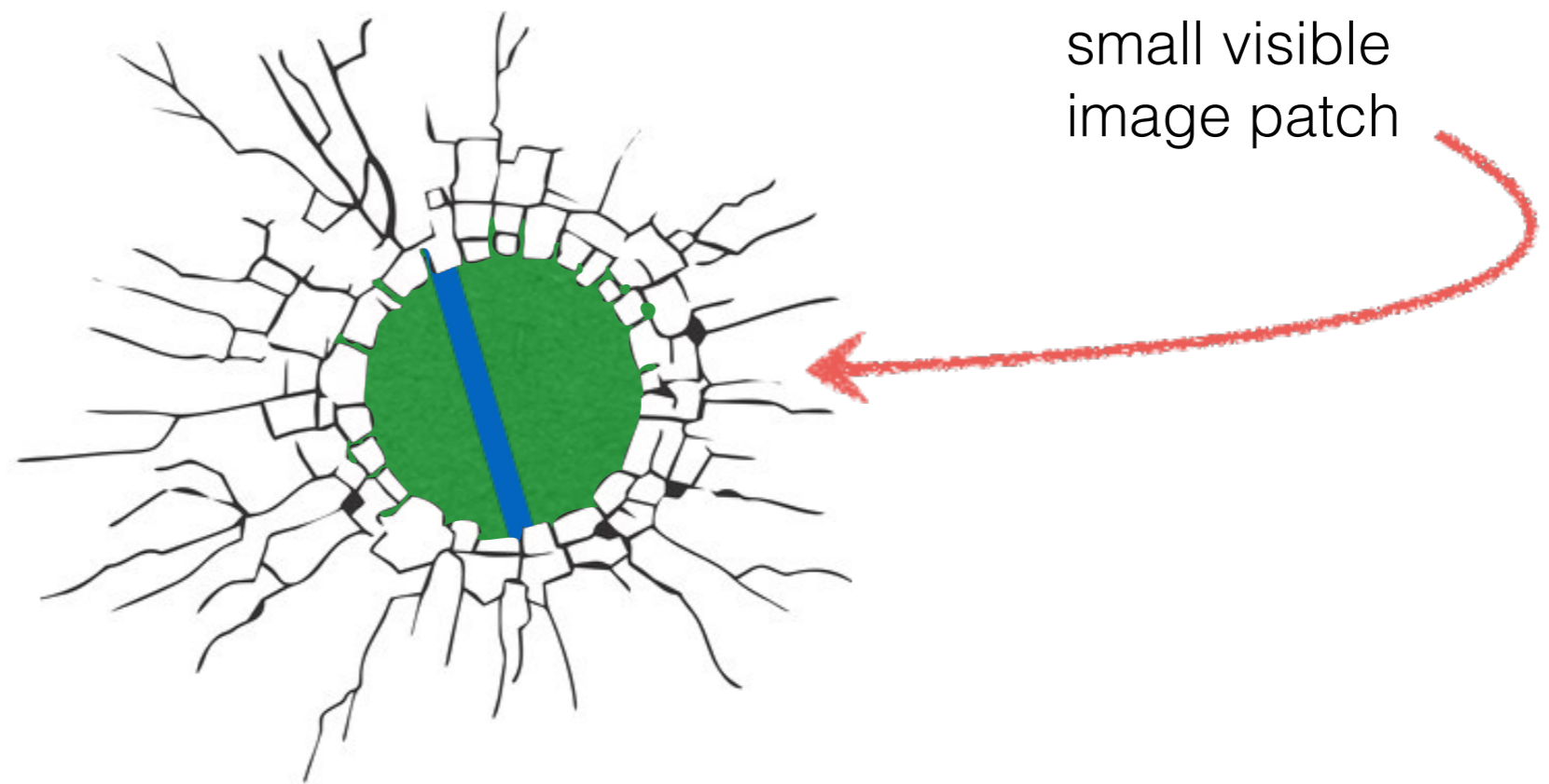
# Barber's pole illusion



# Barber's pole illusion

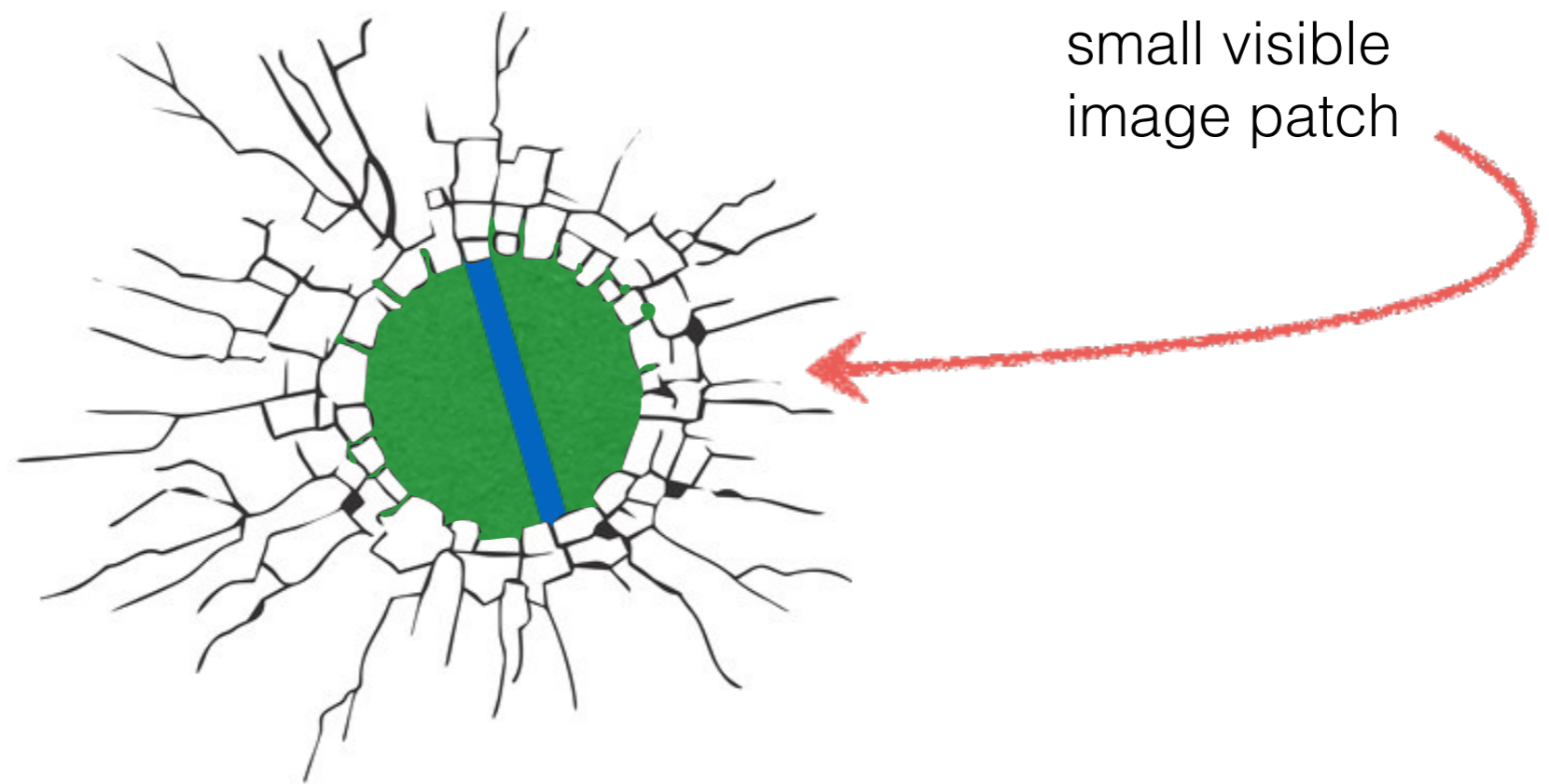


# Aperture Problem



*In which direction is the line moving?*

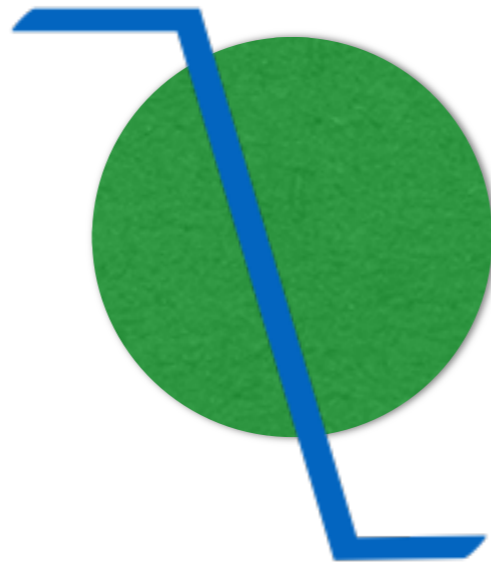
# Aperture Problem



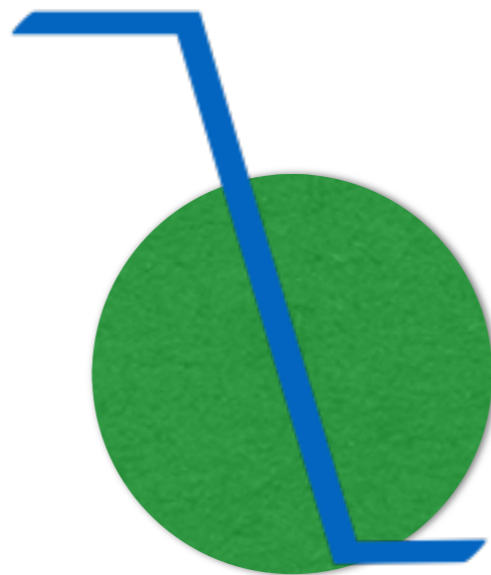
*In which direction is the line moving?*



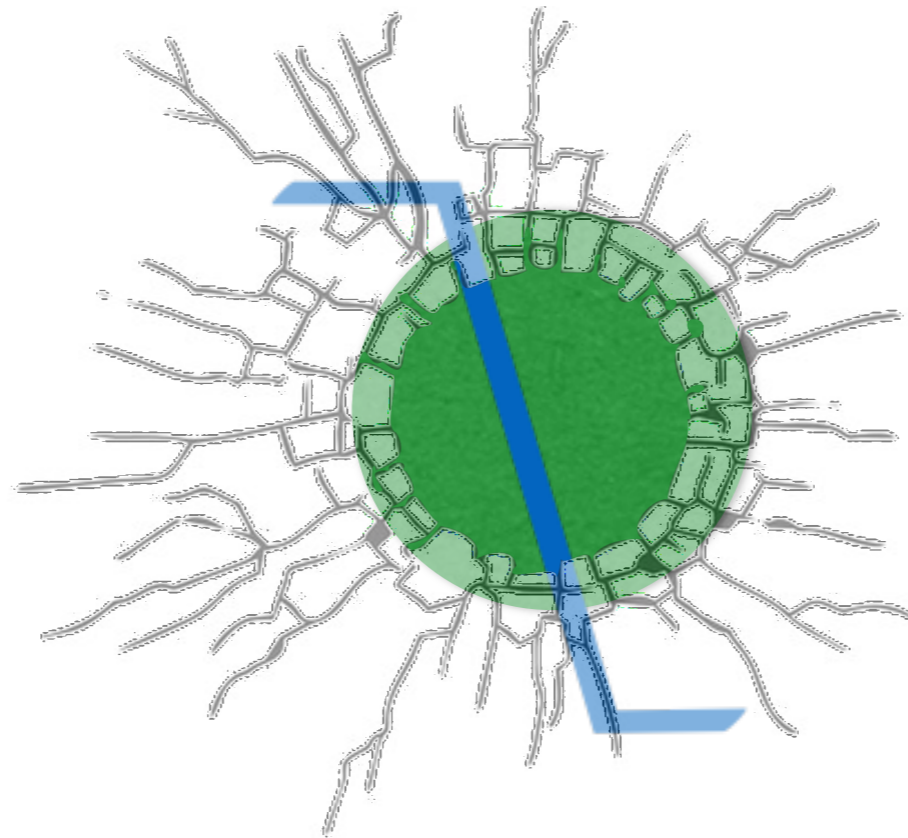
# Aperture Problem



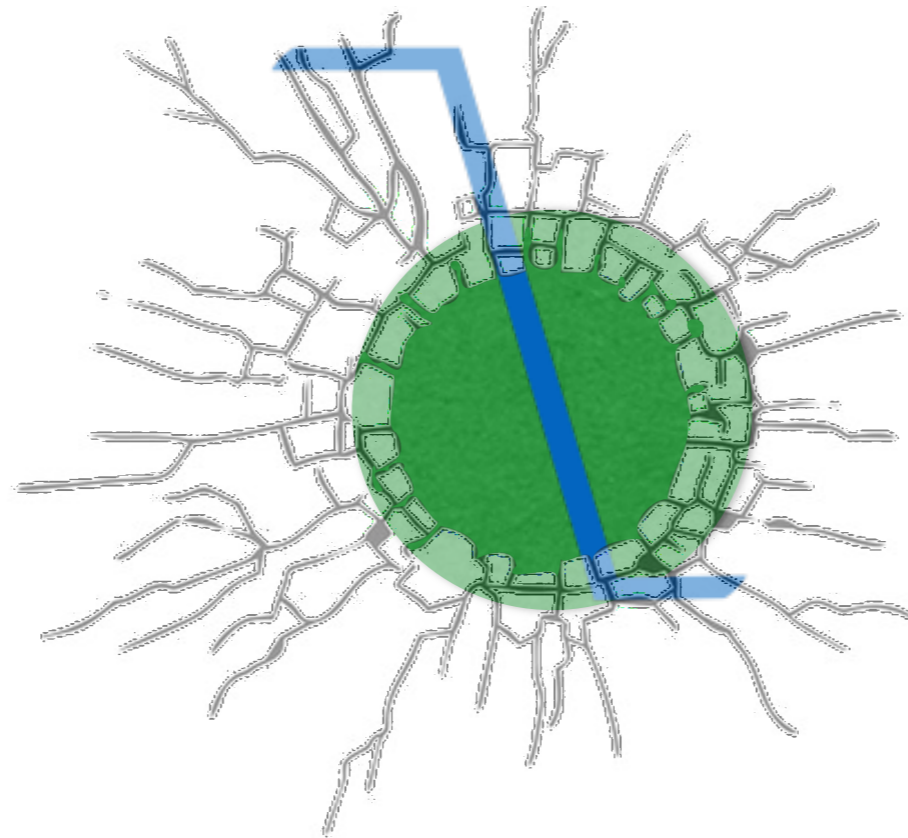
# Aperture Problem

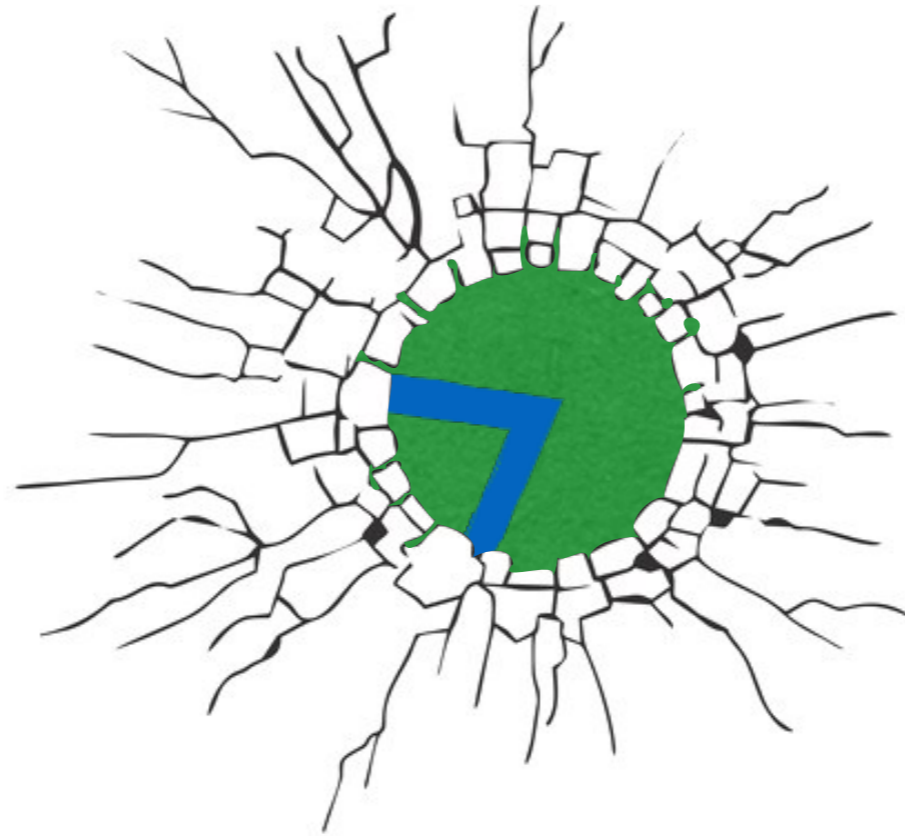


# Aperture Problem

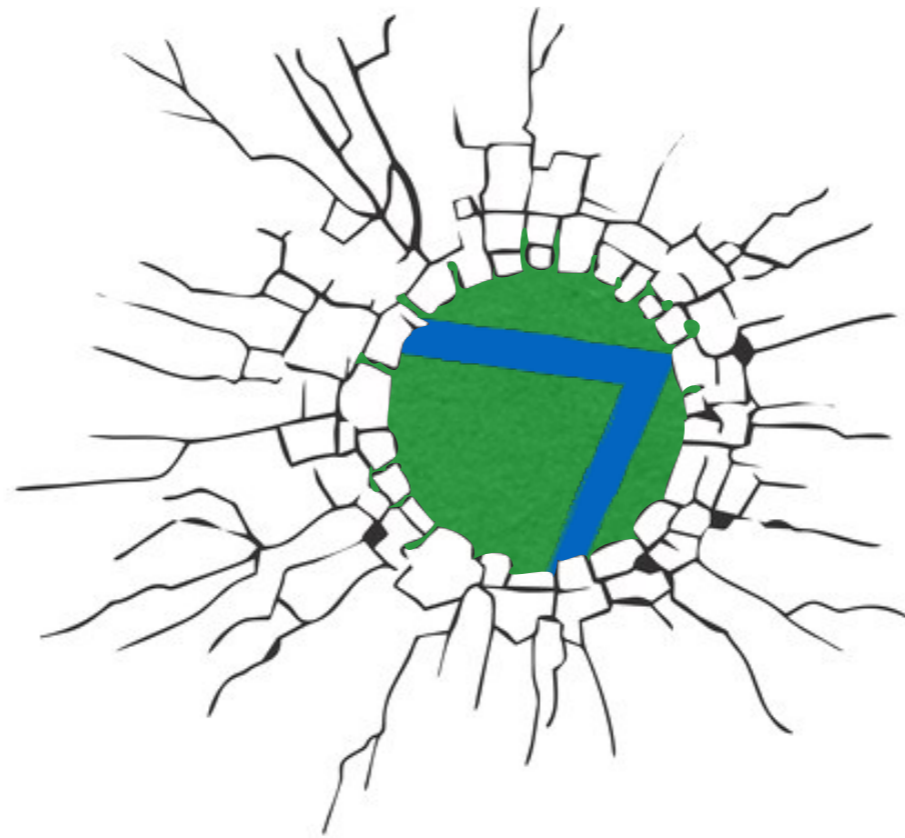


# Aperture Problem

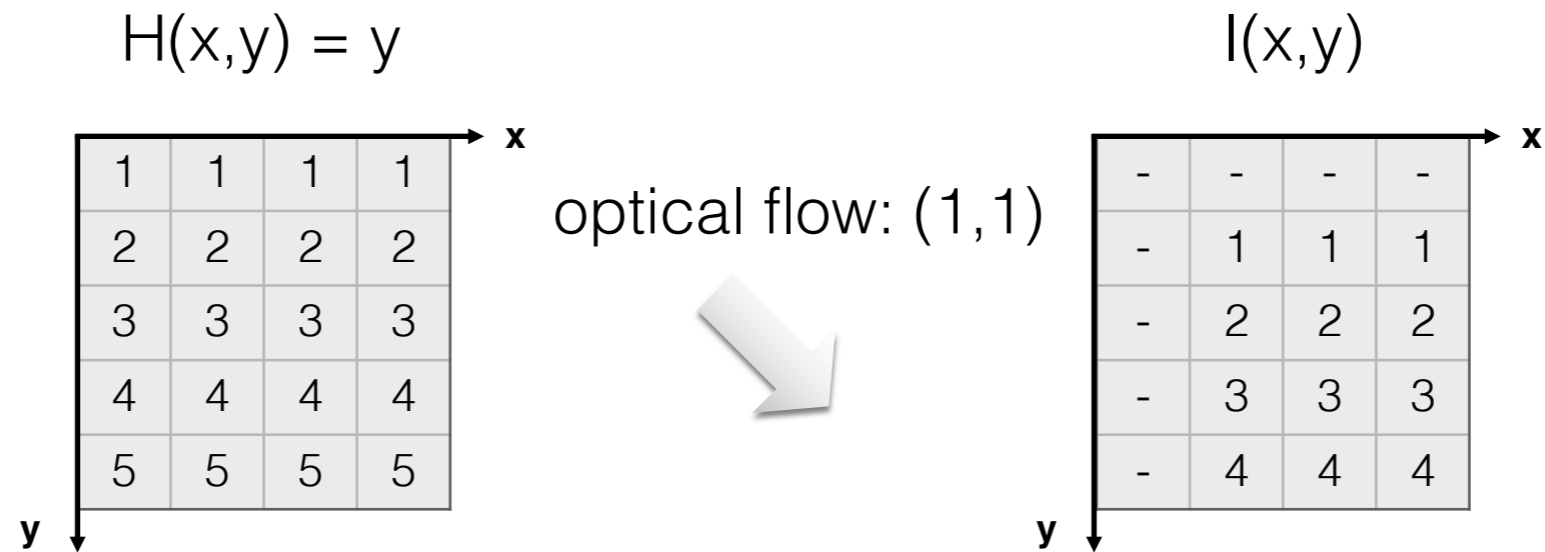




Want patches with different gradients to  
the avoid aperture problem



Want patches with different gradients to  
the avoid aperture problem



$$I_x u + I_y v + I_t = 0$$

**Compute gradients**

$$I_x(3,3) = 0$$

$$I_y(3,3) = 1$$

$$I_t(3,3) = I(3,3) - H(3,3) = -1$$

**Solution:**



$$v = 1$$

We recover the  $v$  of the optical flow but not the  $u$ .

***This is the aperture problem.***

# Horn-Schunck optical flow



## **Horn-Schunck Optical Flow (1981)**

**‘smooth’ flow**

(flow can vary from pixel to pixel)

global method  
(dense)

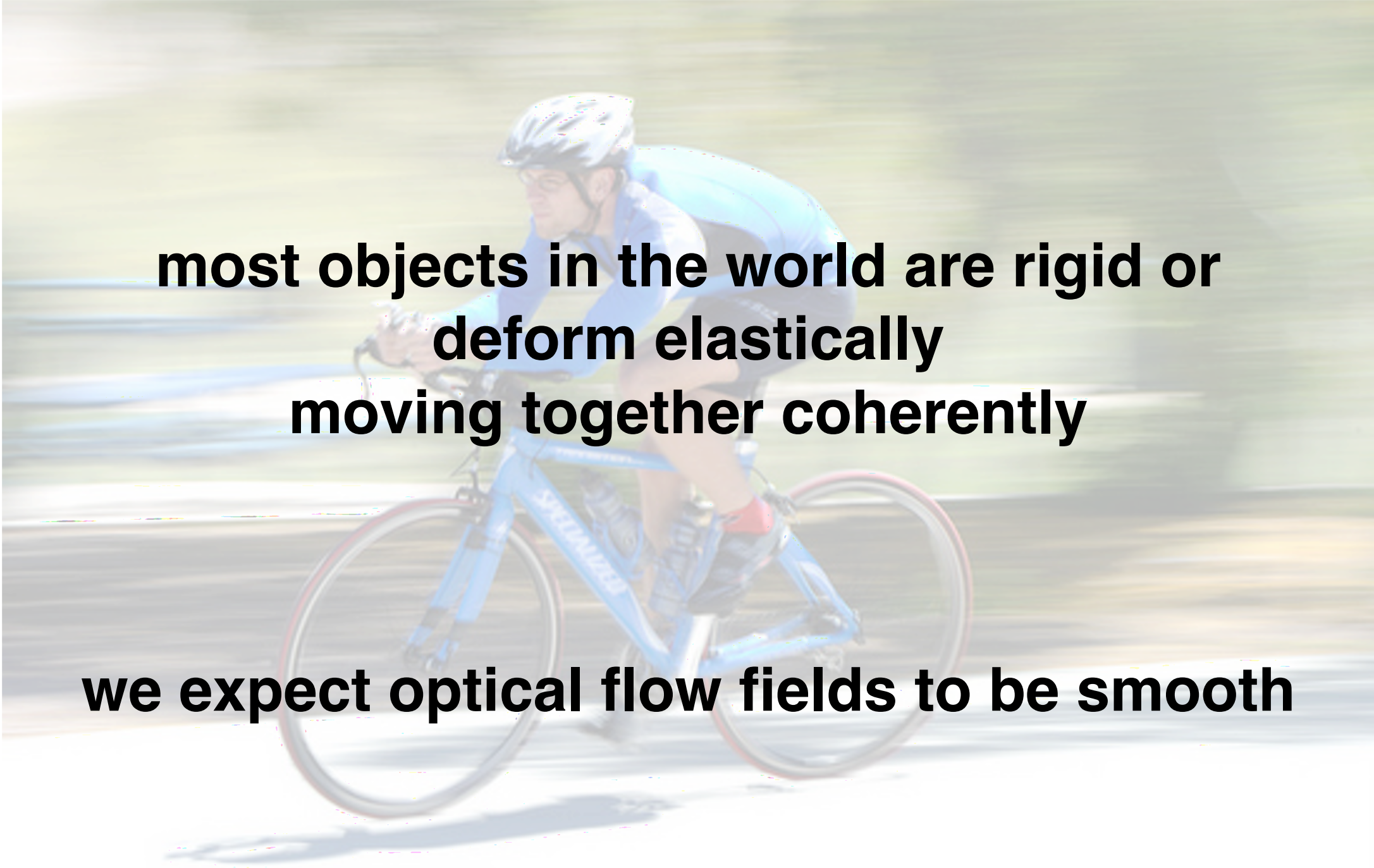
## **Lucas-Kanade Optical Flow (1981)**

**‘constant’ flow**

(flow is constant for all pixels)

local method  
(sparse)

# Smoothness



**most objects in the world are rigid or  
deform elastically  
moving together coherently**

**we expect optical flow fields to be smooth**

# Key idea

(of Horn-Schunck optical flow)

Enforce

**brightness constancy**

Enforce

**smooth flow field**

to compute optical flow

# Key idea

(of Horn-Schunck optical flow)

Enforce

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Enforce

**smooth flow field**

to compute optical flow

# Enforce **brightness constancy**

$$I_x u + I_y v + I_t = 0$$

For every pixel,

$$\min_{u,v} \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

# Enforce brightness constancy

$$I_x u + I_y v + I_t = 0$$

For every pixel,

$$\min_{u,v} \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

lazy notation for  $I_x(i, j)$

# Key idea

(of Horn-Schunck optical flow)

Enforce

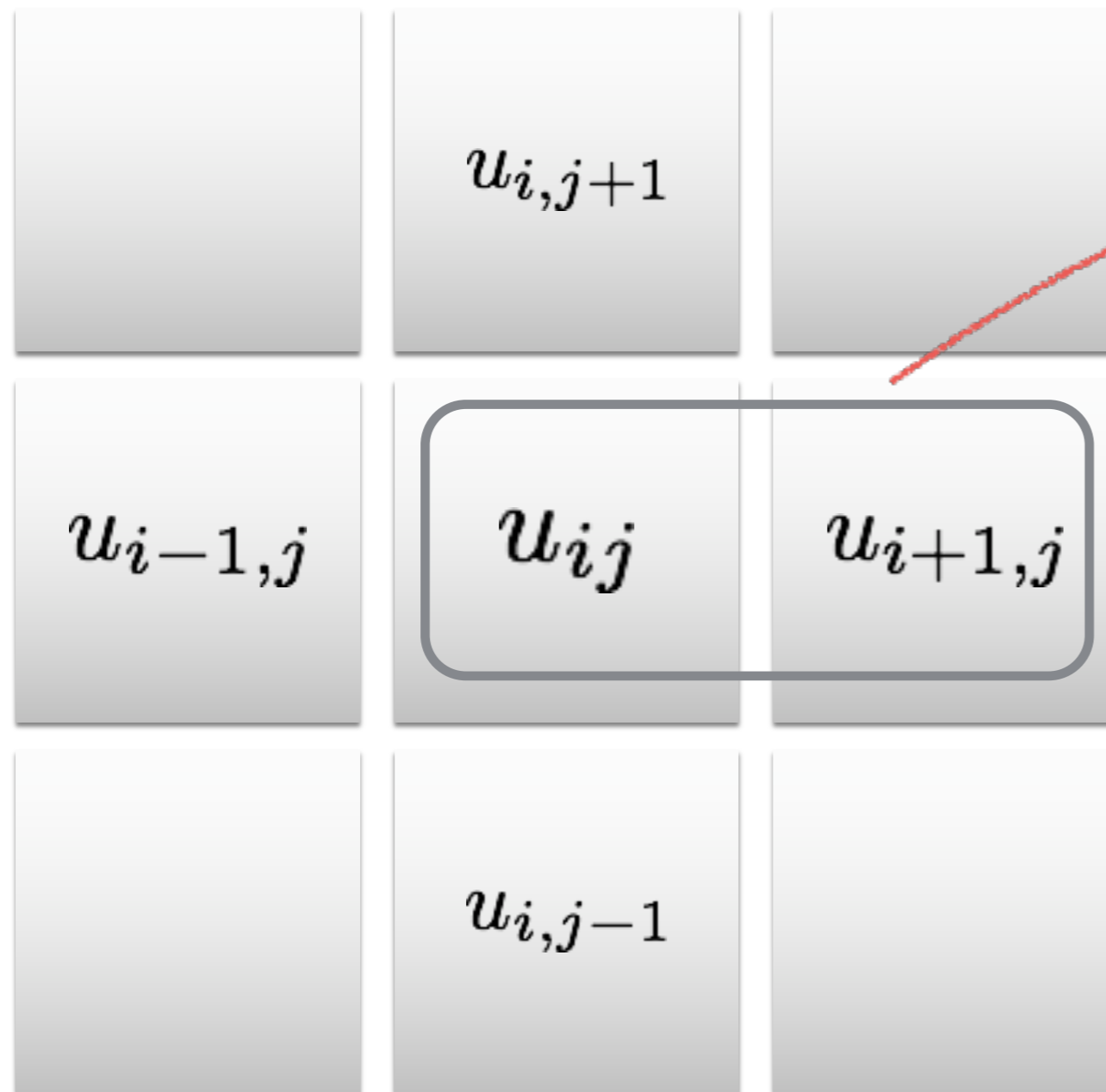
**brightness constancy**

Enforce

**smooth flow field**

to compute optical flow

# Enforce **smooth flow field**

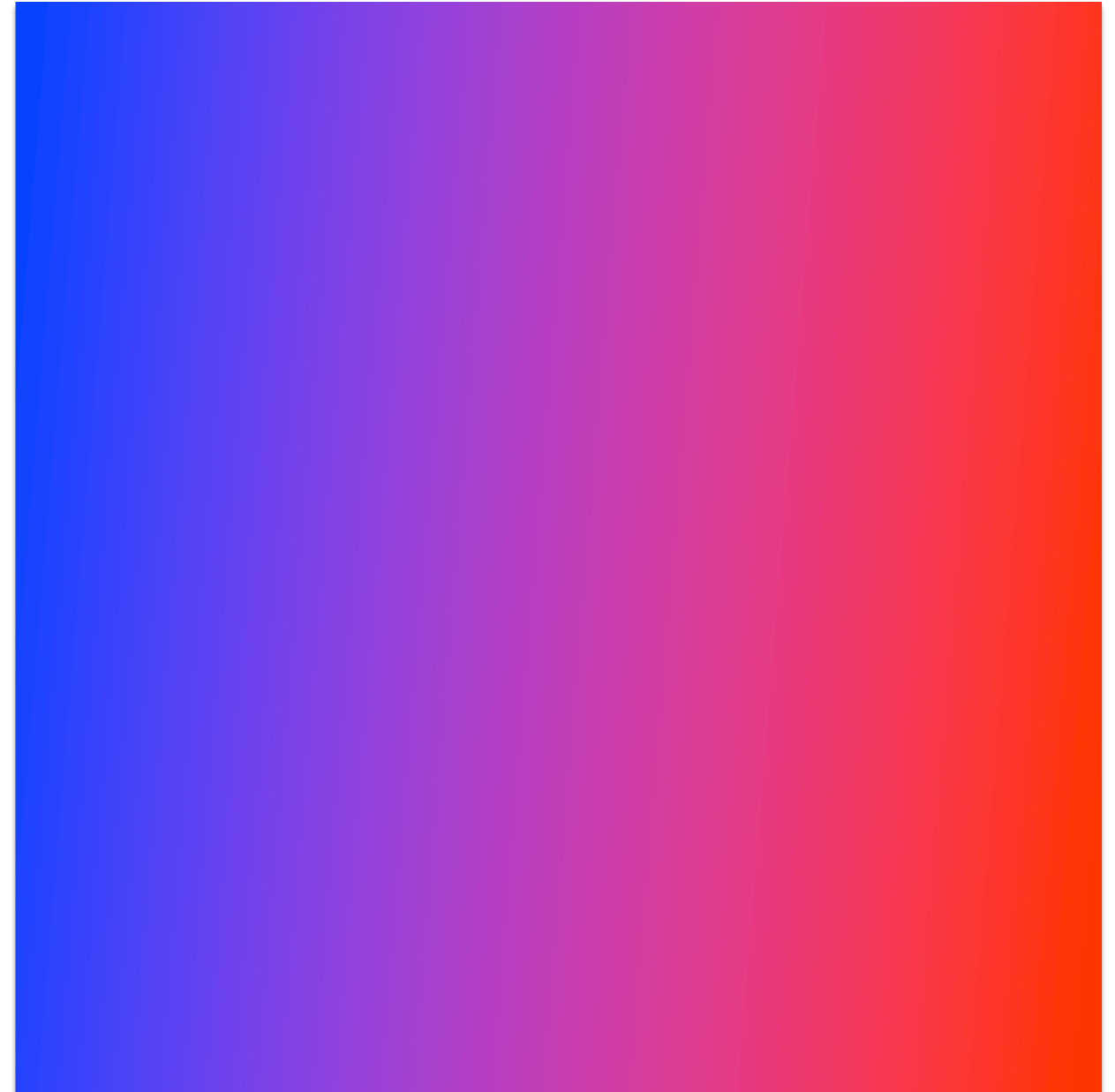
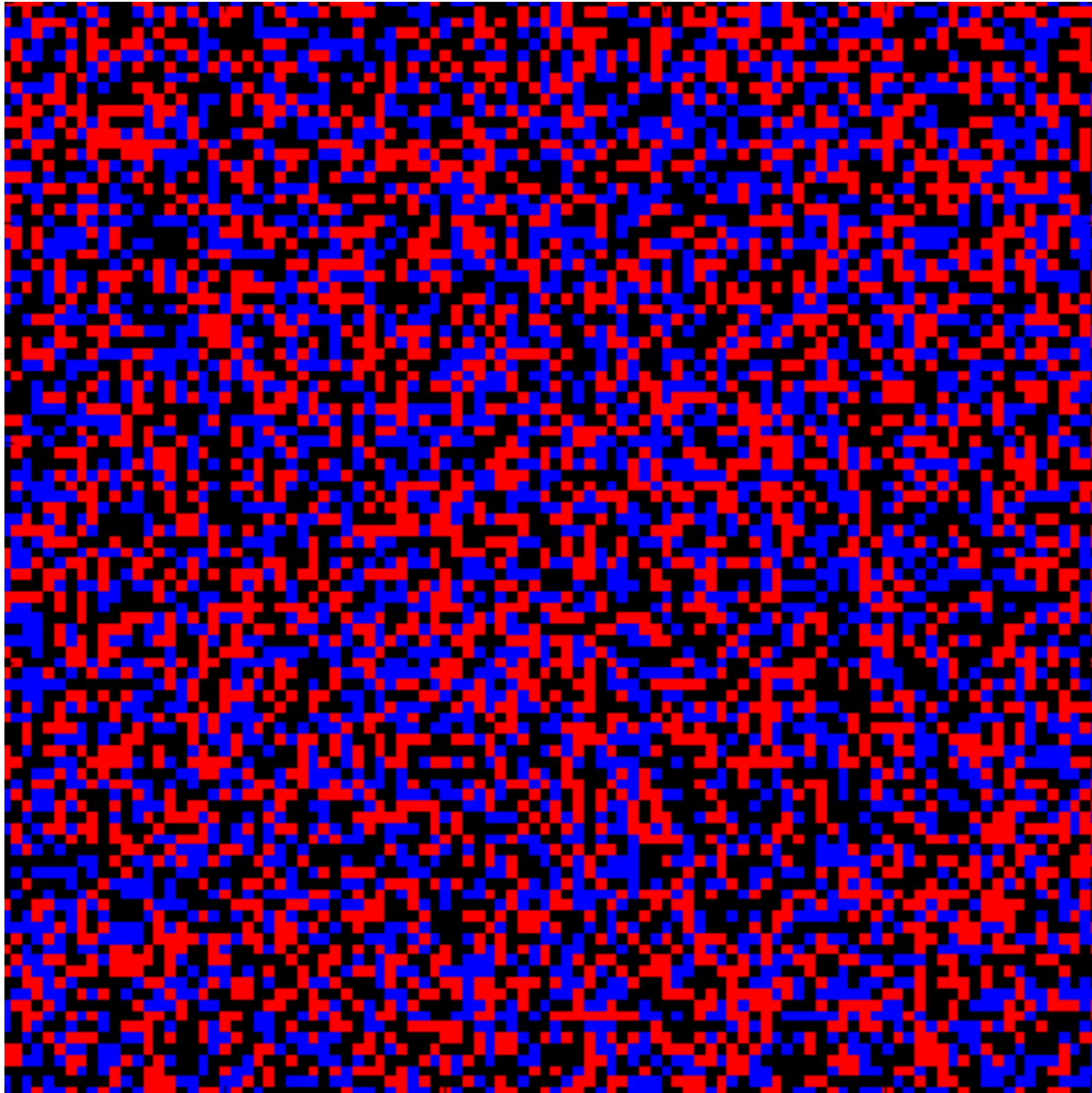


$$\min_{\mathbf{u}} (u_{i,j} - u_{i+1,j})^2$$

u-component of flow



Which flow field optimizes the objective?  $\min_{\mathbf{u}} (u_{i,j} - u_{i+1,j})^2$

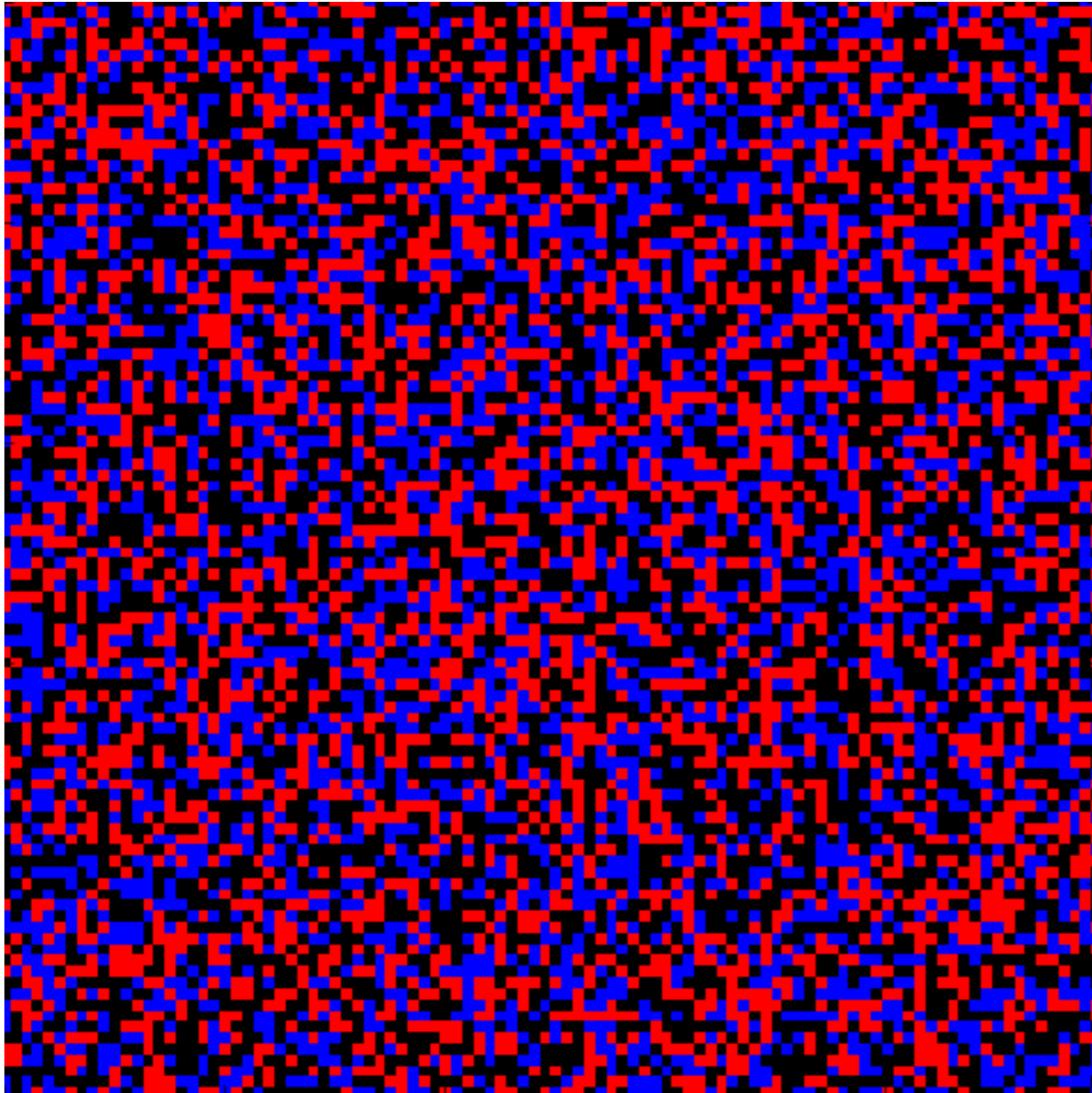


$$\sum_{ij} (u_{ij} - u_{i+1,j})^2$$

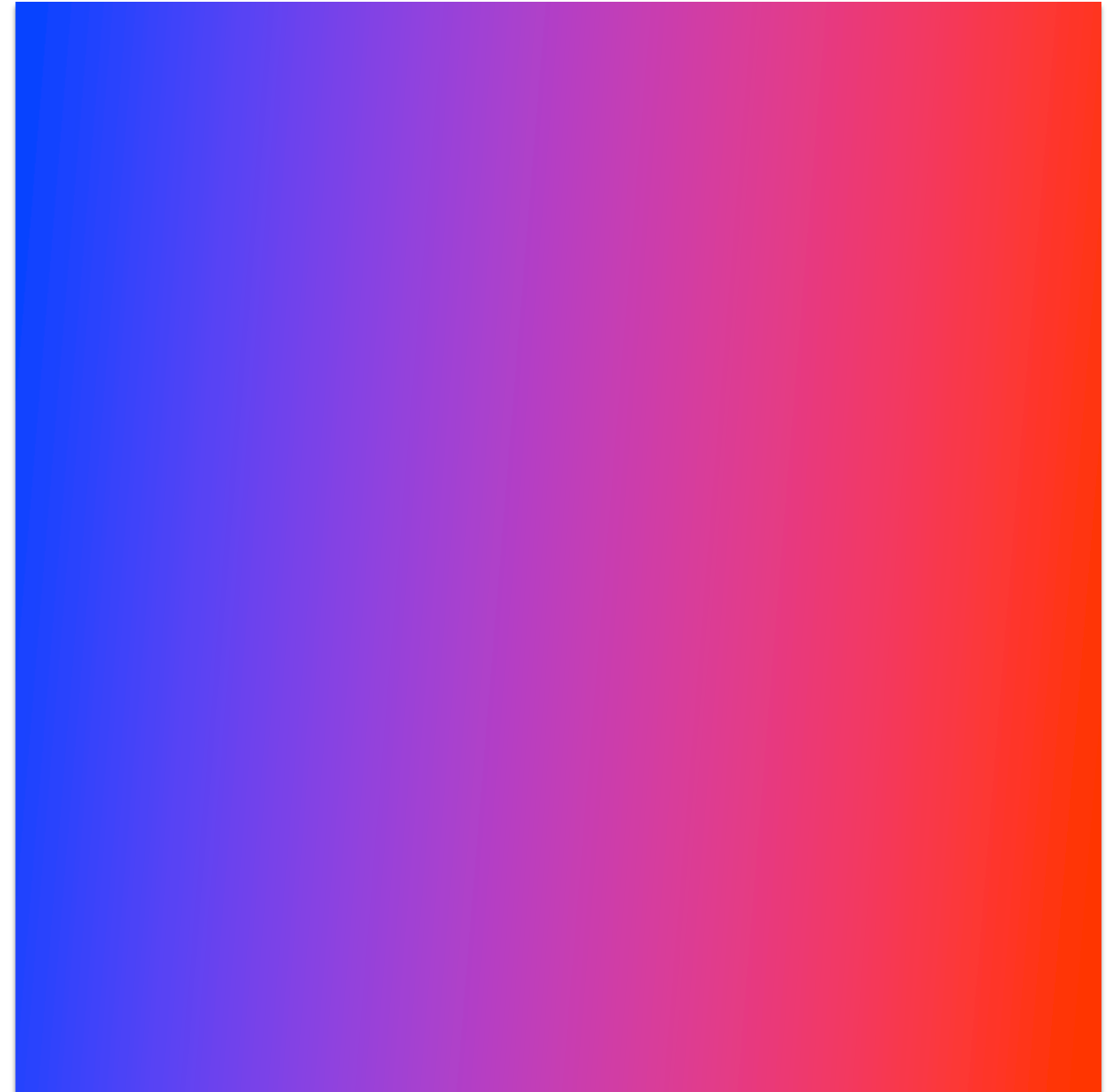
?

$$\sum_{ij} (u_{ij} - u_{i+1,j})^2$$

*Which flow field optimizes the objective?*  $\min_{\mathbf{u}} (u_{i,j} - u_{i+1,j})^2$



big



small

# Key idea

(of Horn-Schunck optical flow)

Enforce

**brightness constancy**

Enforce

**smooth flow field**

to compute optical flow

bringing it all together...

# Horn-Schunck optical flow

$$\min_{\mathbf{u}, \mathbf{v}} \sum_{i, j} \left\{ \begin{array}{l} \text{smoothness} \\ E_s(i, j) \end{array} + \begin{array}{l} \text{brightness constancy} \\ \lambda E_d(i, j) \end{array} \right\}$$

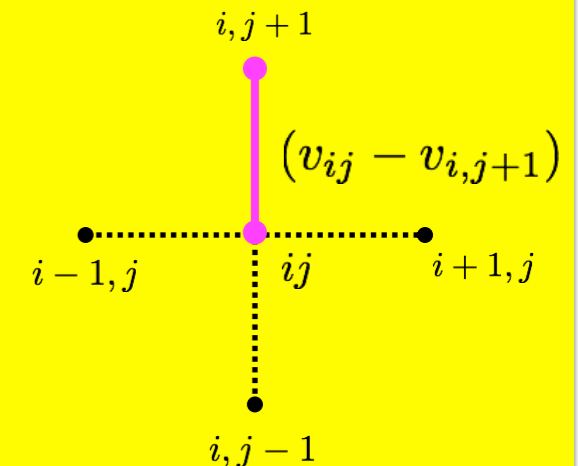
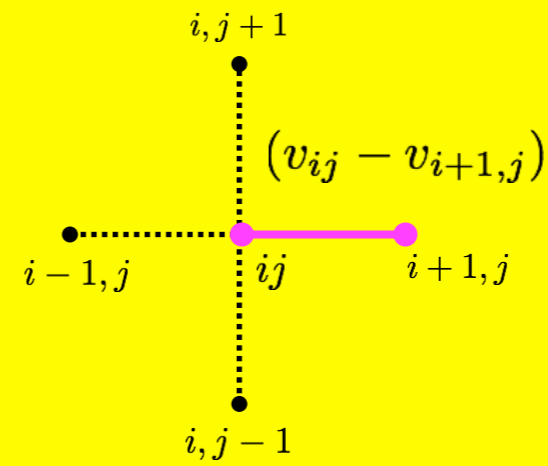
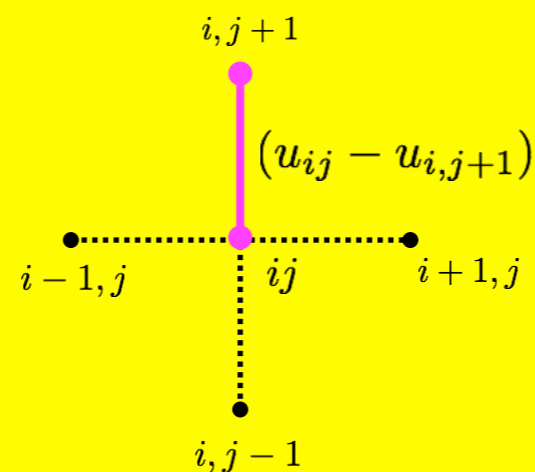
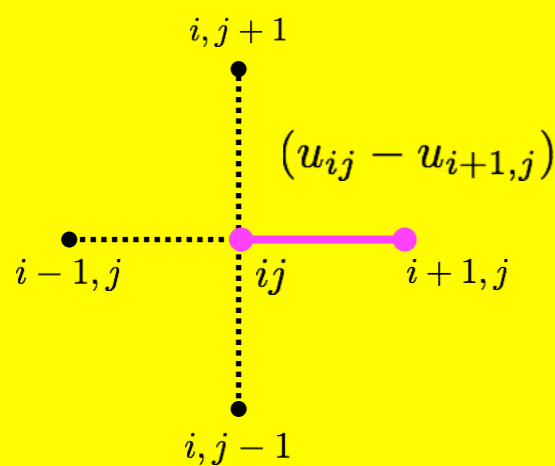
weight

# HS optical flow objective function

**Brightness constancy**  $E_d(i, j) = \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2$

## Smoothness

$$E_s(i, j) = \frac{1}{4} \left[ (u_{ij} - u_{i+1, j})^2 + (u_{ij} - u_{i, j+1})^2 + (v_{ij} - v_{i+1, j})^2 + (v_{ij} - v_{i, j+1})^2 \right]$$



How do we solve this minimization problem?

$$\min_{\mathbf{u}, \mathbf{v}} \sum_{i, j} \left\{ E_s(i, j) + \lambda E_d(i, j) \right\}$$

How do we solve this minimization problem?

$$\min_{\mathbf{u}, \mathbf{v}} \sum_{i, j} \left\{ E_s(i, j) + \lambda E_d(i, j) \right\}$$

Compute partial derivative, derive update equations  
(gradient decent!)

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

smoothness term brightness constancy



Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad...

$$\frac{\partial E}{\partial u_{kl}} =$$

*how many u terms depend on k and l?*

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad...

$$\frac{\partial E}{\partial u_{kl}} =$$

*how many u terms depend on k and l?*

**FOUR** from smoothness

**ONE** from brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad...

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

*how many u terms depend on k and l?*

**FOUR** from smoothness

**ONE** from brightness constancy

Compute the partial derivatives of this huge sum!

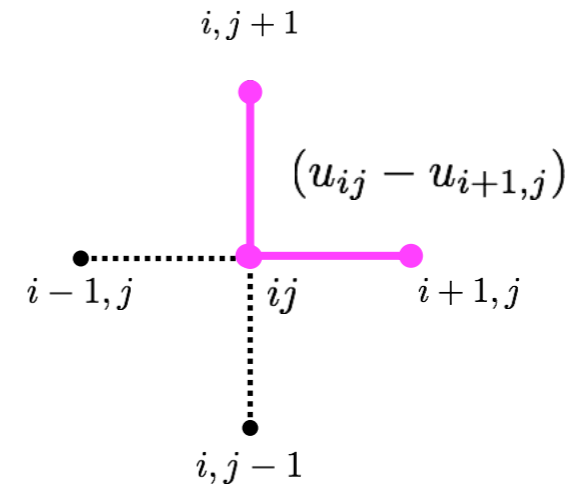
$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$



$$(u_{ij}^2 - 2u_{ij}u_{i+1,j} + u_{i+1,j}^2)$$

$$(u_{ij}^2 - 2u_{ij}u_{i,j+1} + u_{i,j+1}^2)$$

(variable will appear four times in sum)



Compute the partial derivatives of this huge sum!

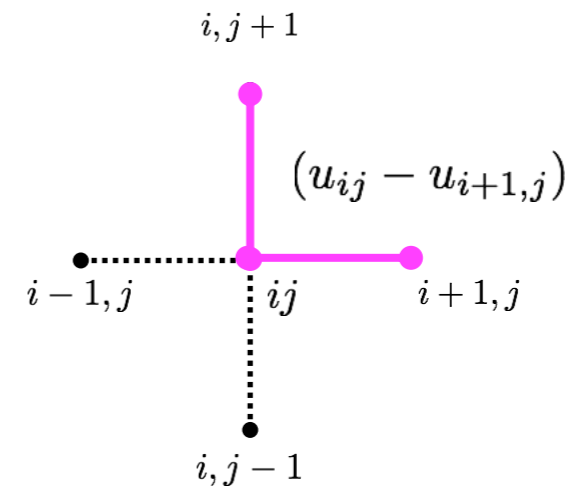
$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$



$$(u_{ij}^2 - 2u_{ij}u_{i+1,j} + u_{i+1,j}^2)$$

$$(u_{ij}^2 - 2u_{ij}u_{i,j+1} + u_{i,j+1}^2)$$

(variable will appear four times in sum)



$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

short hand for  
local average

$$\bar{u}_{ij} = \frac{1}{4} \left\{ u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} \right\}$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

*Where are the extrema of E?*

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

*Where are the extrema of E?*

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

*Where are the extrema of E?*

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

*this is a linear system*       **$\mathbf{Ax} = \mathbf{b}$**       *how do you solve this?*



$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

Recall  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\text{adj}\mathbf{A}}{\det \mathbf{A}}\mathbf{b}$

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

Recall  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\text{adj}\mathbf{A}}{\det \mathbf{A}}\mathbf{b}$

Same as the linear system:

$$\underbrace{\{1 + \lambda(I_x^2 + I_y^2)\}}_{(\det A)} u_{kl} = (1 + \lambda I_y^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

$$\underbrace{\{1 + \lambda(I_x^2 + I_y^2)\}}_{(\det A)} v_{kl} = (1 + \lambda I_x^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}u_{kl} = (1 + \lambda I_y^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}v_{kl} = (1 + \lambda I_x^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

Rearrange to get update equations:

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value
old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

**Recall:**  $\min_{\mathbf{u}, \mathbf{v}} \sum_{i,j} \left\{ E_s(i, j) + \lambda E_d(i, j) \right\}$

When lambda is small (lambda inverse is big)...

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value
old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

**Recall:**  $\min_{\mathbf{u}, \mathbf{v}} \sum_{i,j} \left\{ E_s(i, j) + \lambda E_d(i, j) \right\}$

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new value
old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

goes to zero
goes to zero

**Recall:**  $\min_{\mathbf{u}, \mathbf{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$

When lambda is small (lambda inverse is big)...

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value      old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

goes to zero

goes to zero

...we only care about smoothness.

ok, take a step back, why did we do all this math?

We are solving for the optical flow (u,v) given two constraints

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

smoothness brightness constancy

We needed the math to minimize this  
(now to the algorithm)



# Horn-Schunck Optical Flow Algorithm

1. Precompute image gradients  $I_y$   $I_x$
2. Precompute temporal gradients  $I_t$
3. Initialize flow field  $u = 0$   
 $v = 0$
4. While not converged

Compute flow field updates for each pixel:

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \quad \hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

**Just 8 lines of code!**