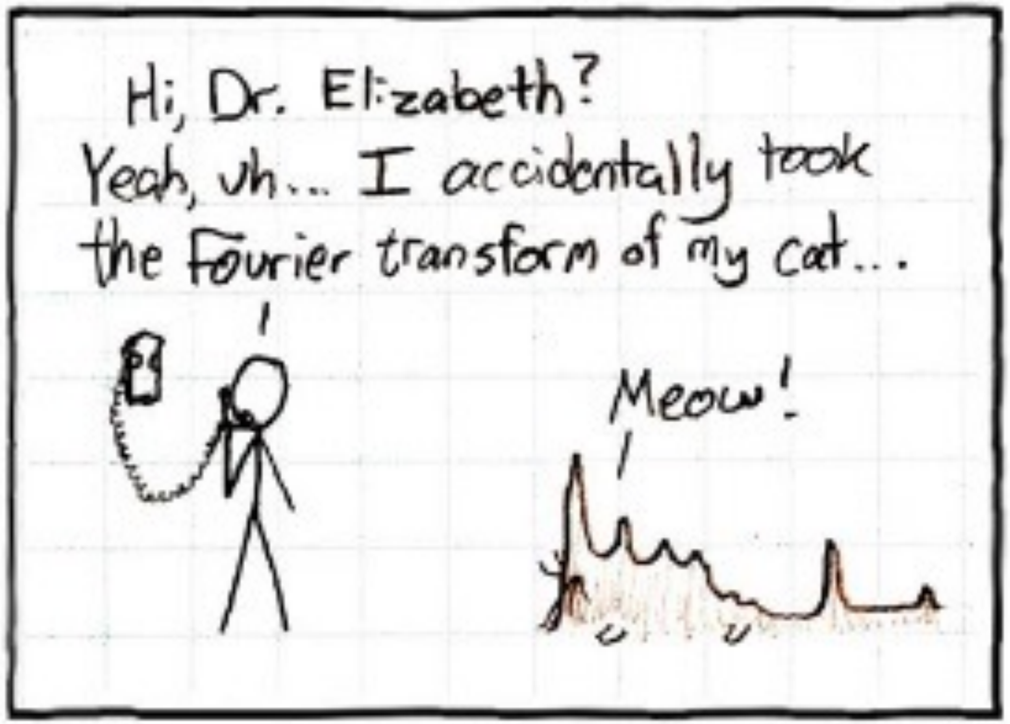
Image pyramids and frequency domain



16-385 Computer Vision
Spring 2024, Lecture 3

http://16385.courses.cs.cmu.edu/

Overview of today's lecture

- Image downsampling.
- Aliasing.
- Gaussian image pyramid.
- Laplacian image pyramid.
- Fourier series.
- Frequency domain.
- Fourier transform.
- Frequency-domain filtering.
- Revisiting sampling.

Slide credits

Most of these slides were adapted directly from:

- Kris Kitani (15-463, Fall 2016).

Some slides were inspired or taken from:

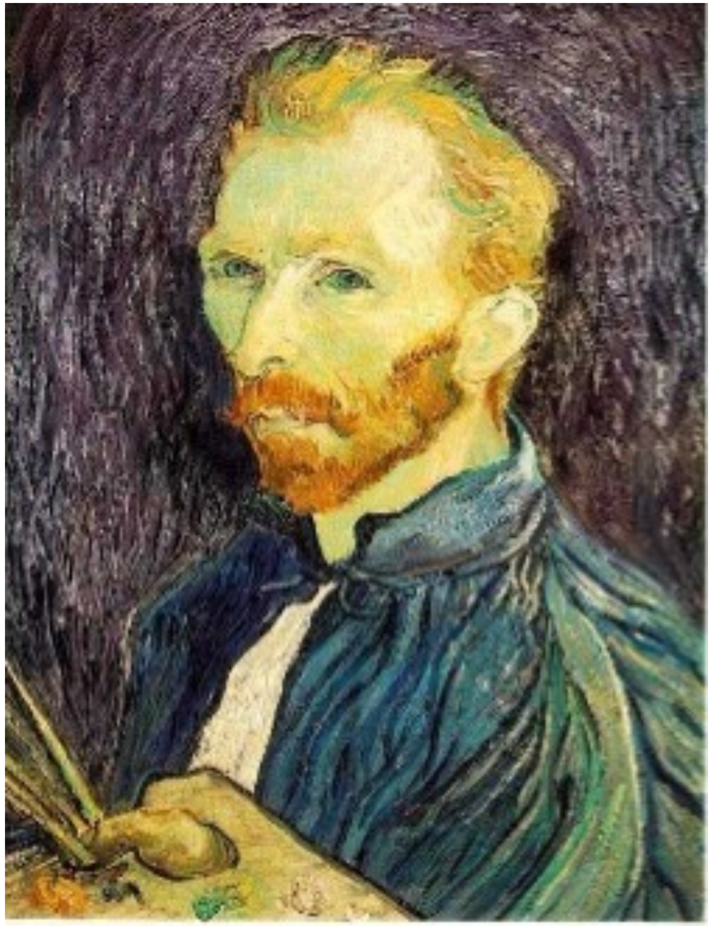
- Fredo Durand (MIT).
- Bernd Girod (Stanford University).
- James Hays (Georgia Tech).
- Steve Marschner (Cornell University).
- Steve Seitz (University of Washington).

Image downsampling

This image is too big to fit on the screen.
How would you reduce it to half its size?

Naïve image downsampling

Throw away half the rows and columns



1/2

delete even rows
delete even columns

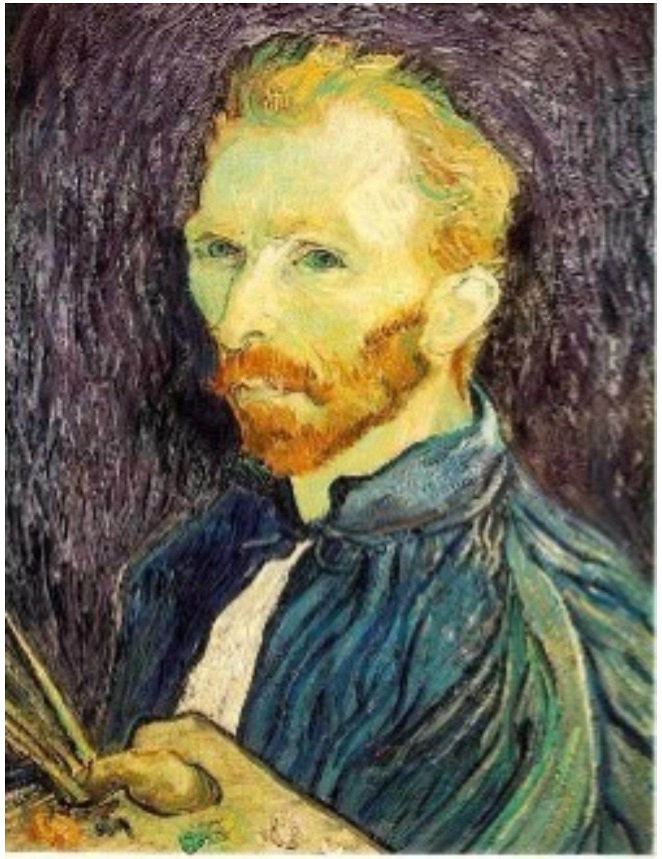
1/4

delete even rows
delete even columns

1/8

What is the problem with this approach?

Naïve image downsampling



1/2

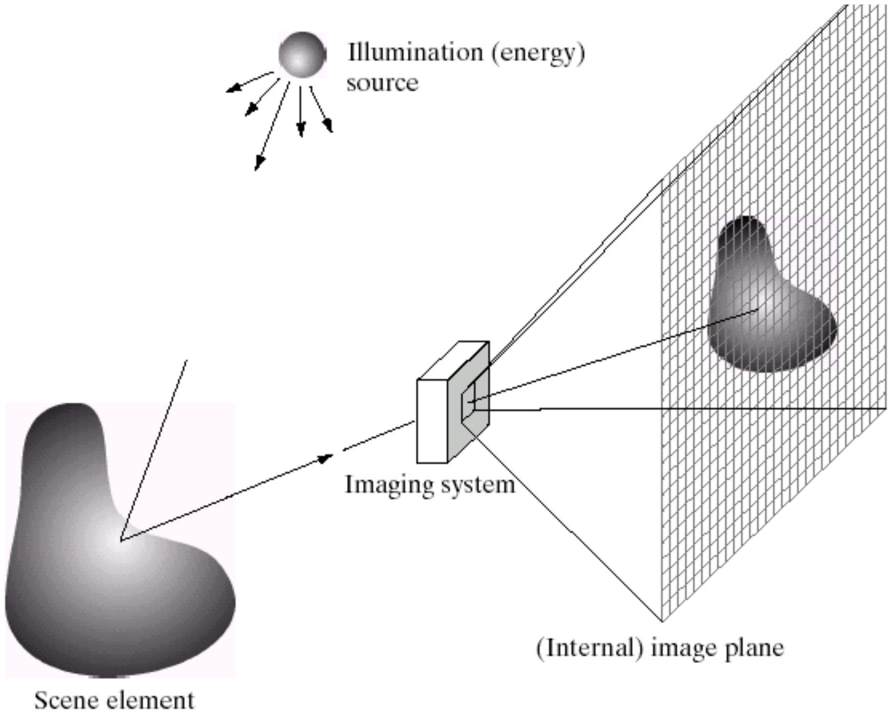
1/4 (2x zoom)

1/8 (4x zoom)

Why is the 1/8 image so pixelated (and do you know what this effect is called)?

Aliasing

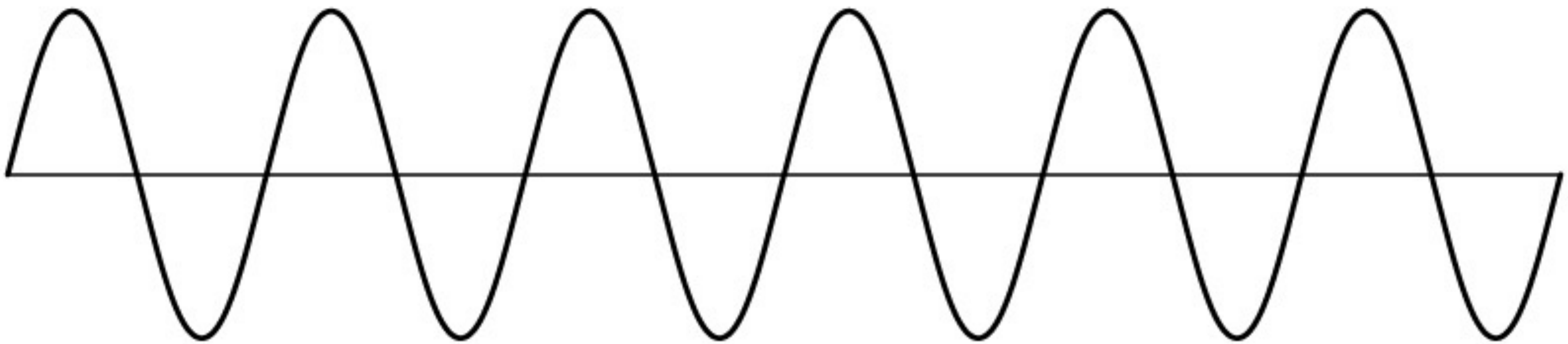
Reminder



Images are a *discrete*, or *sampled*, representation of a *continuous* world

Sampling

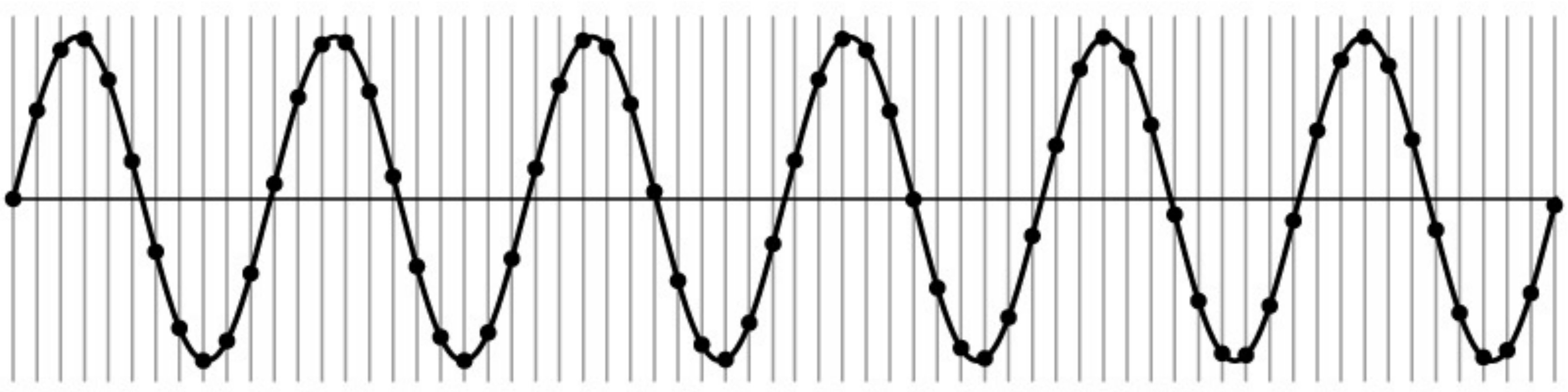
Very simple example: a sine wave



How would you discretize this signal?

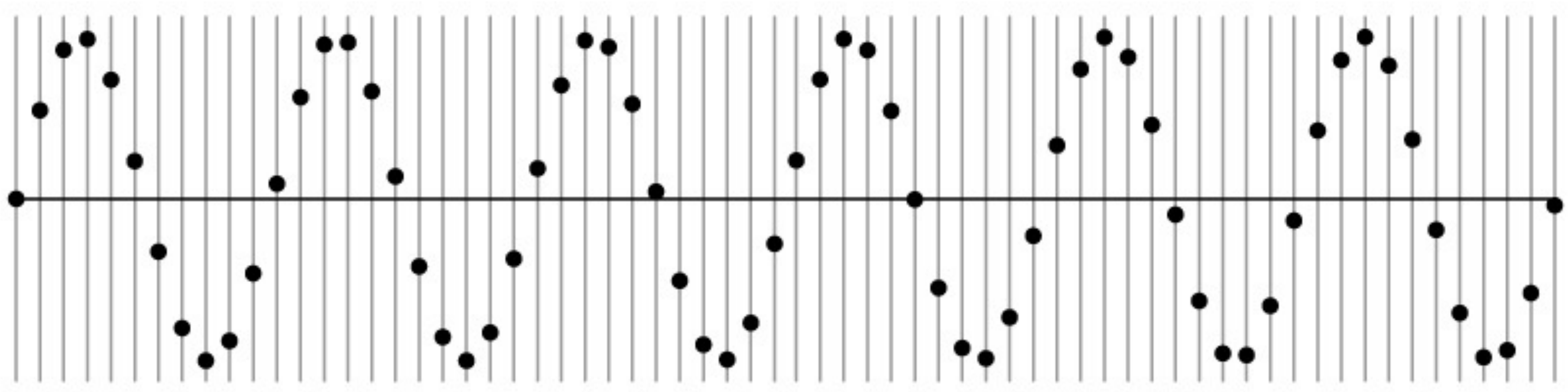
Sampling

Very simple example: a sine wave



Sampling

Very simple example: a sine wave

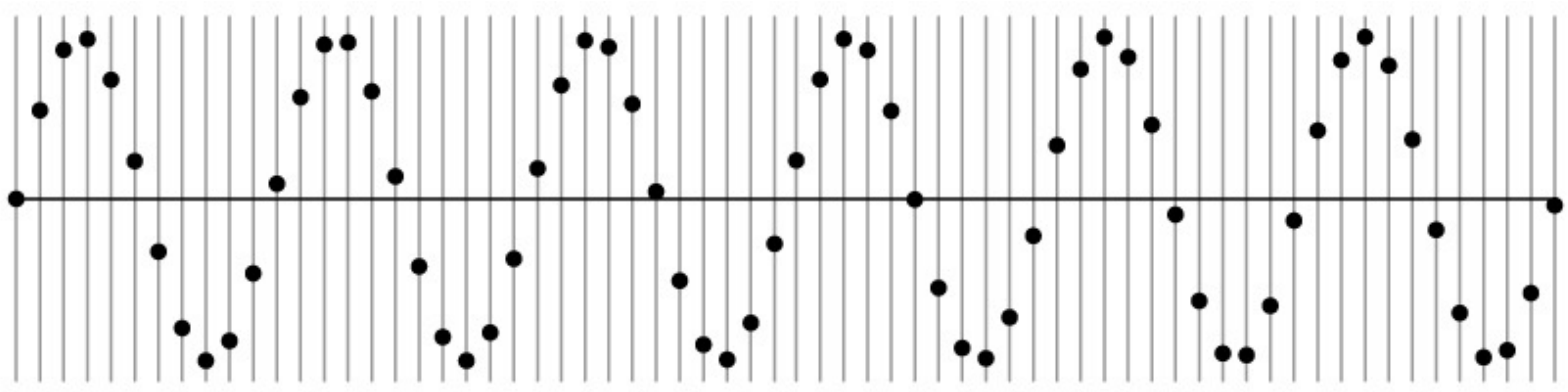


How many samples should I take?

Can I take as *many* samples as I want?

Sampling

Very simple example: a sine wave

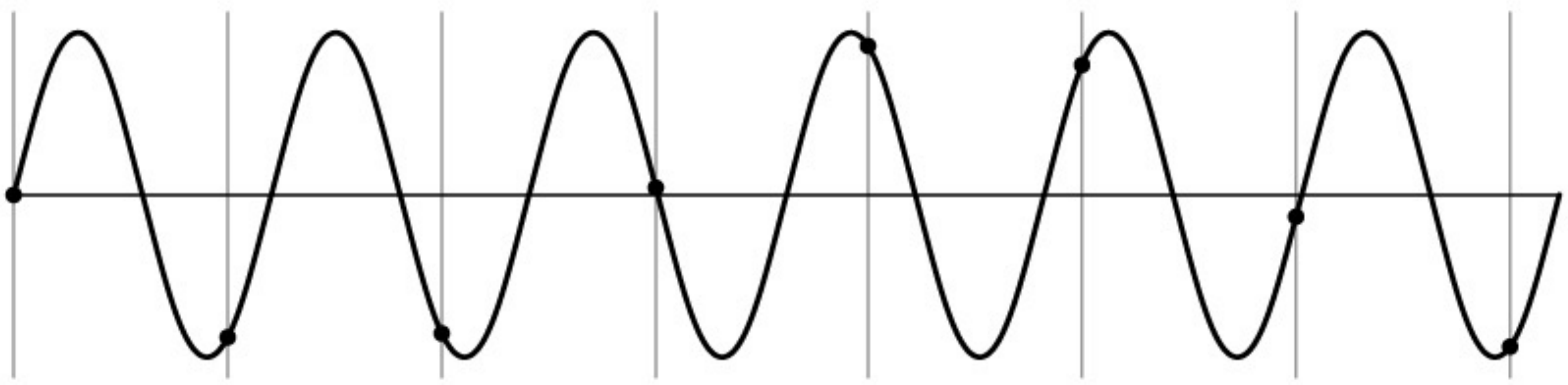


How many samples should I take?

Can I take as *few* samples as I want?

Undersampling

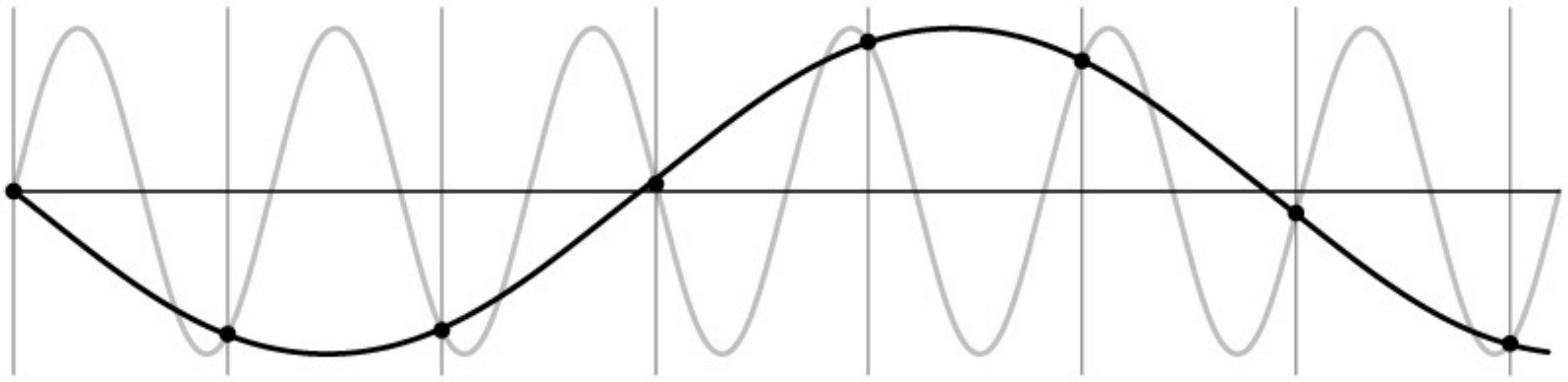
Very simple example: a sine wave



Unsurprising effect: information is lost.

Undersampling

Very simple example: a sine wave

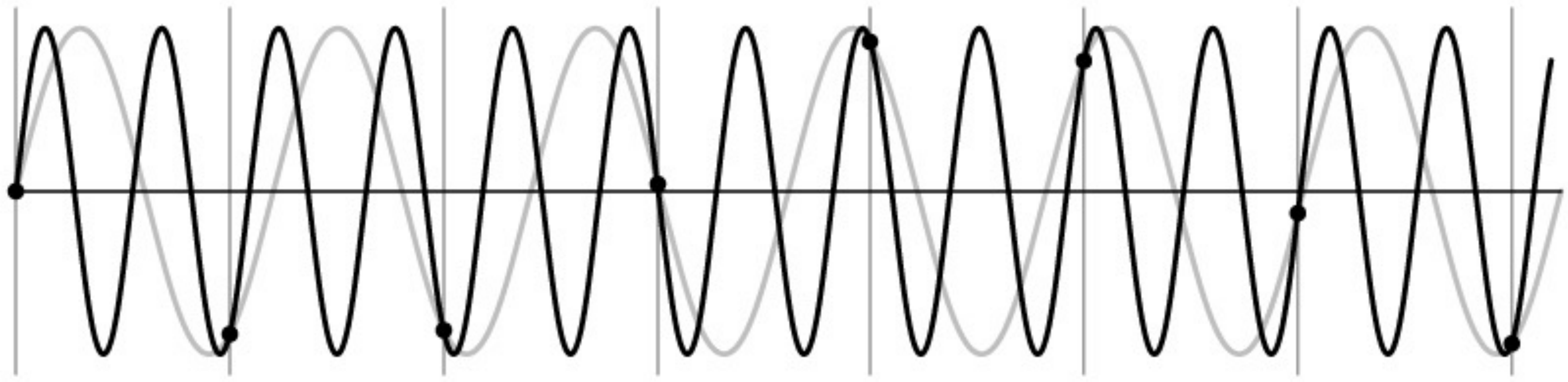


Unsurprising effect: information is lost.

Surprising effect: can confuse the signal with one of *lower* frequency.

Undersampling

Very simple example: a sine wave



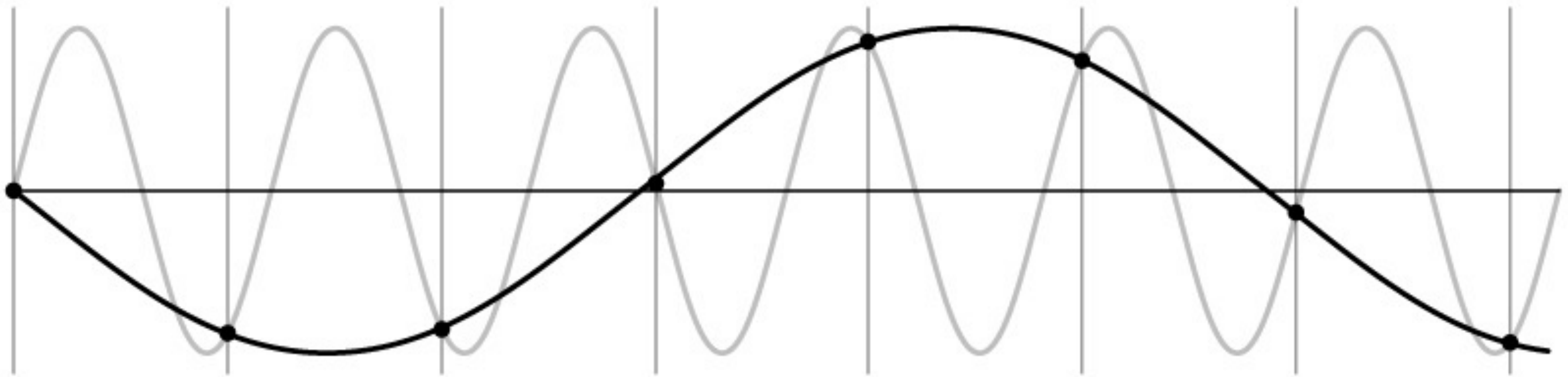
Unsurprising effect: information is lost.

Surprising effect: can confuse the signal with one of *lower* frequency.

Note: we could always confuse the signal with one of *higher* frequency.

Aliasing

Fancy term for: *Undersampling can disguise a signal as one of a lower frequency*

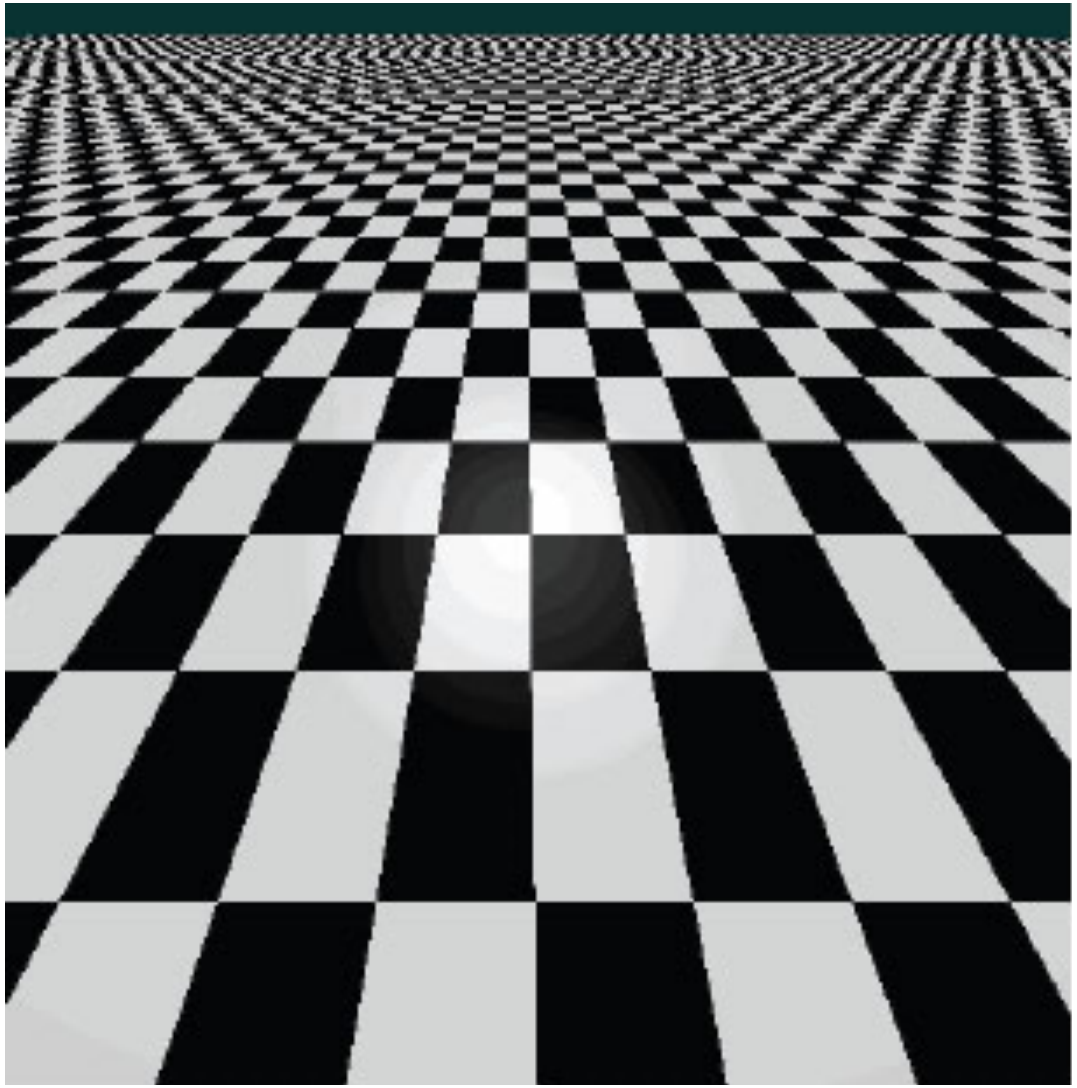


Unsurprising effect: information is lost.

Surprising effect: can confuse the signal with one of *lower* frequency.

Note: we could always confuse the signal with one of *higher* frequency.

Aliasing in textures



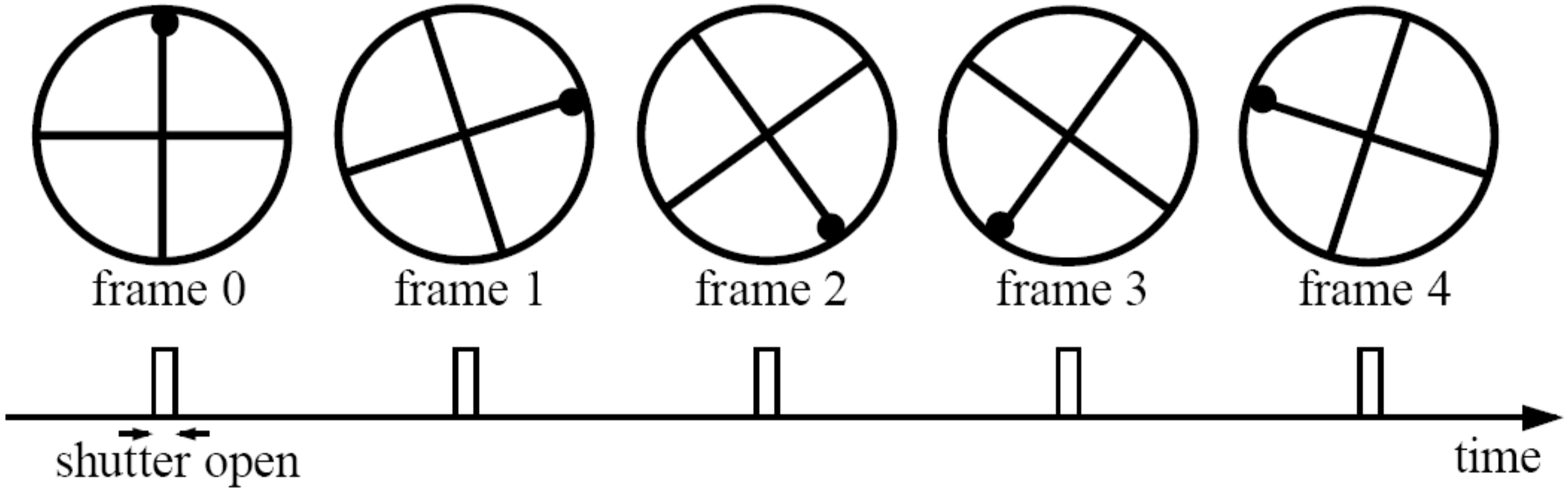
Aliasing in photographs

This is also known as "moire"

Temporal aliasing

Imagine a spoked wheel moving to the right (rotating clockwise). Mark wheel with dot so we can see what's happening.

If camera shutter is only open for a fraction of a frame time (frame time = 1/30 sec. for video, 1/24 sec. for film):



Without dot, wheel appears to be rotating slowly backwards! (counterclockwise)

Wagon wheel effect

Anti-aliasing

How would you deal with aliasing?

Anti-aliasing

How would you deal with aliasing?

Approach 1: Oversample the signal

Anti-aliasing

How would you deal with aliasing?

Approach 1: Oversample the signal

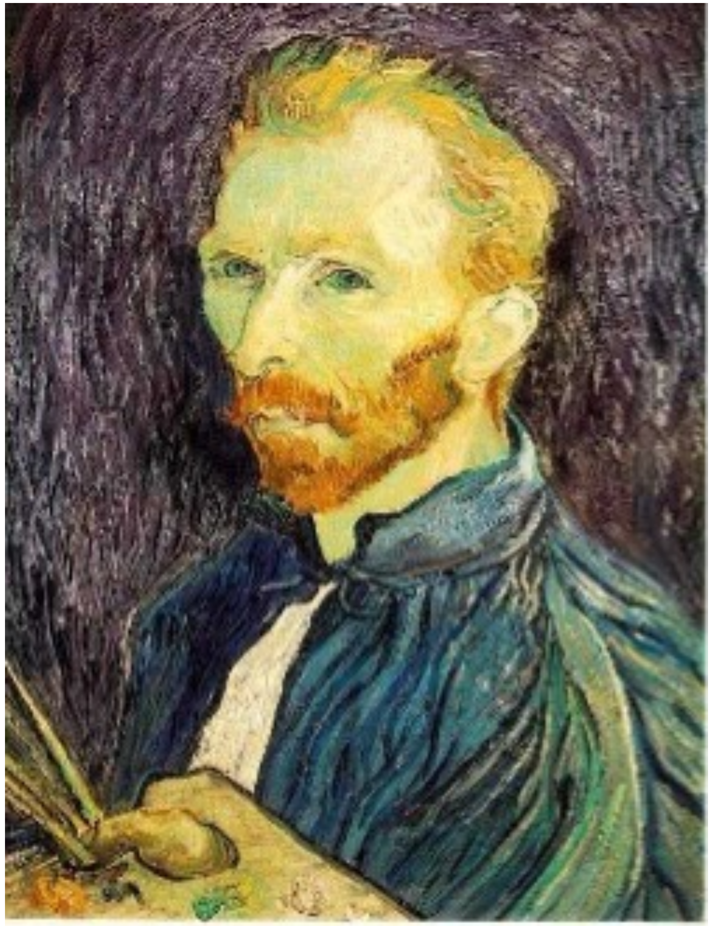
Approach 2: Smooth the signal

- Remove some of the detail effects that cause aliasing.
- Lose information, but better than aliasing artifacts.

How would you smooth a signal?

Better image downsampling

Apply a smoothing filter first, then throw away half the rows and columns



1/2

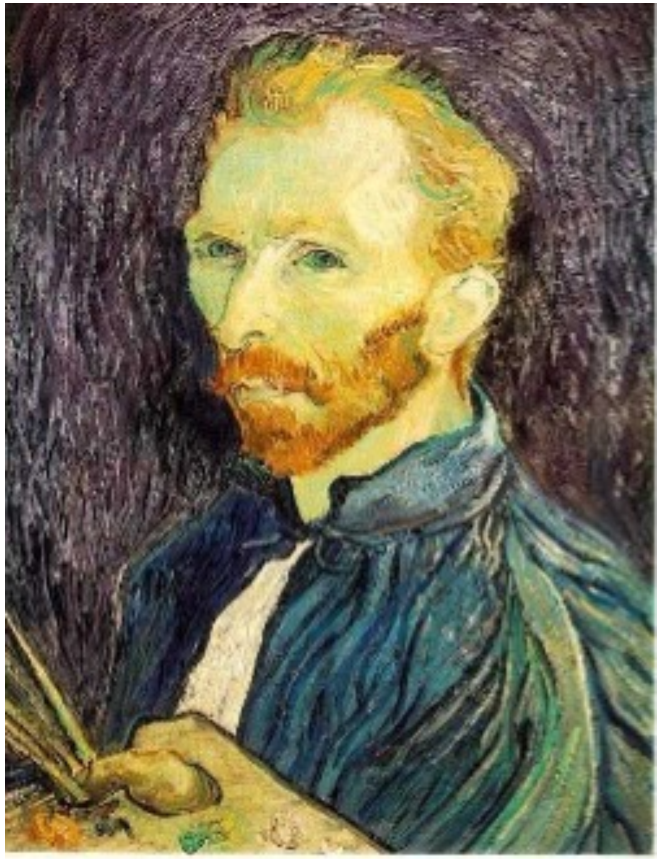
Gaussian filter
delete even rows
delete even columns

1/4

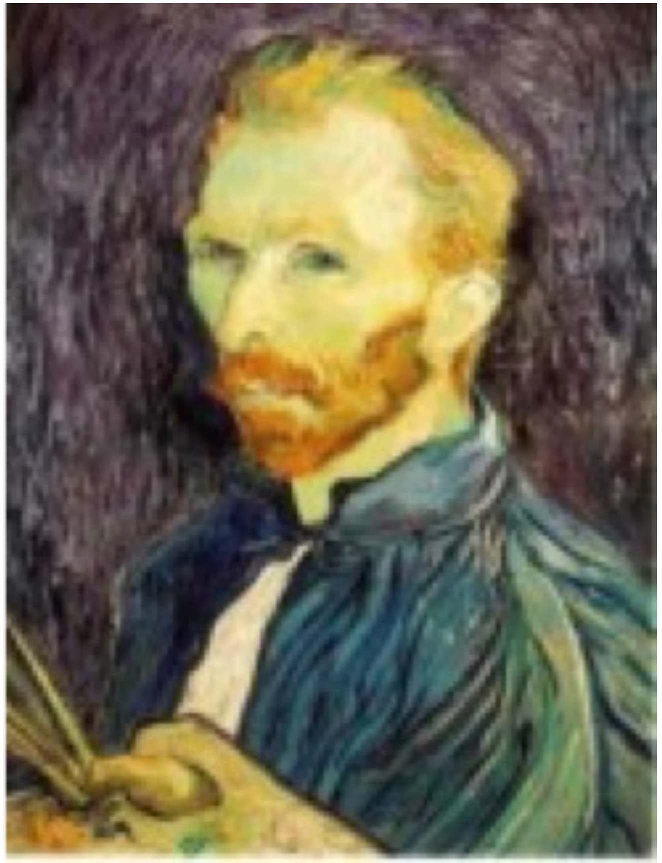
Gaussian filter
delete even rows
delete even columns

1/8

Better image downsampling



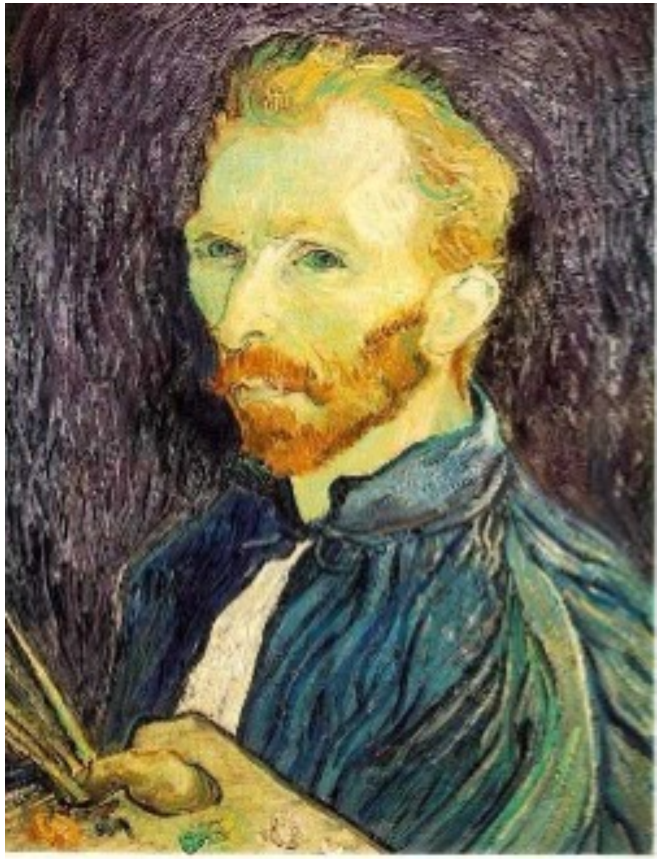
1/2



1/4 (2x zoom)

1/8 (4x zoom)

Naïve image downsampling



1/2

1/4 (2x zoom)

1/8 (4x zoom)

Anti-aliasing

Question 1: How much smoothing is needed to avoid aliasing?

Question 2: How many samples are needed to avoid aliasing?

Answer to both: Enough to reach the Nyquist limit. (We’ll see what this means soon.)

Gaussian image pyramid

Gaussian image pyramid

The name of this sequence of subsampled images

Constructing a Gaussian pyramid

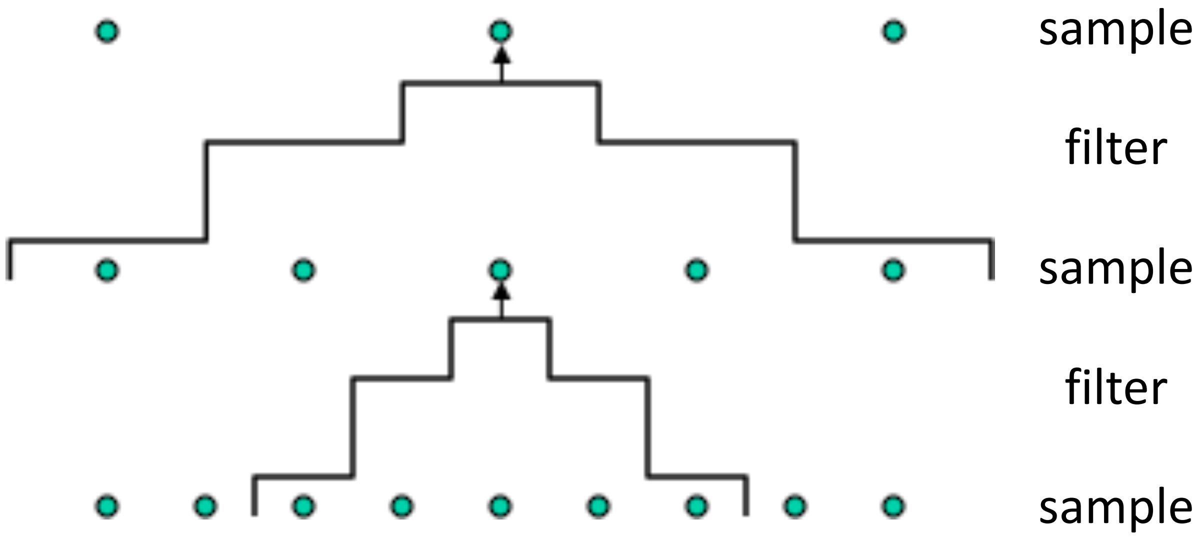
Algorithm

repeat:

filter

subsample

until min resolution reached



Question: How much bigger than the original image is the whole pyramid?

Constructing a Gaussian pyramid

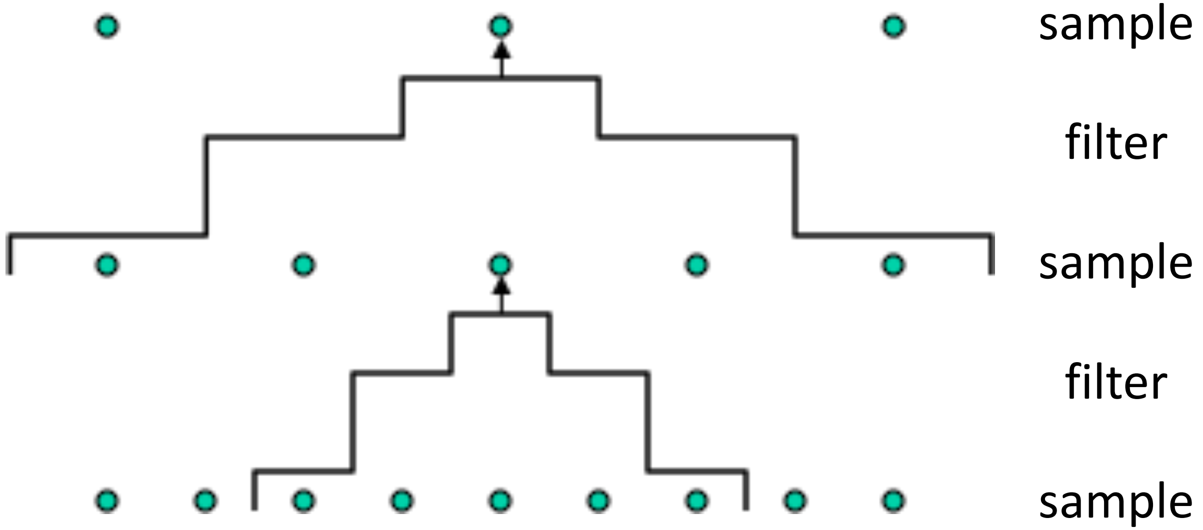
Algorithm

repeat:

filter

subsample

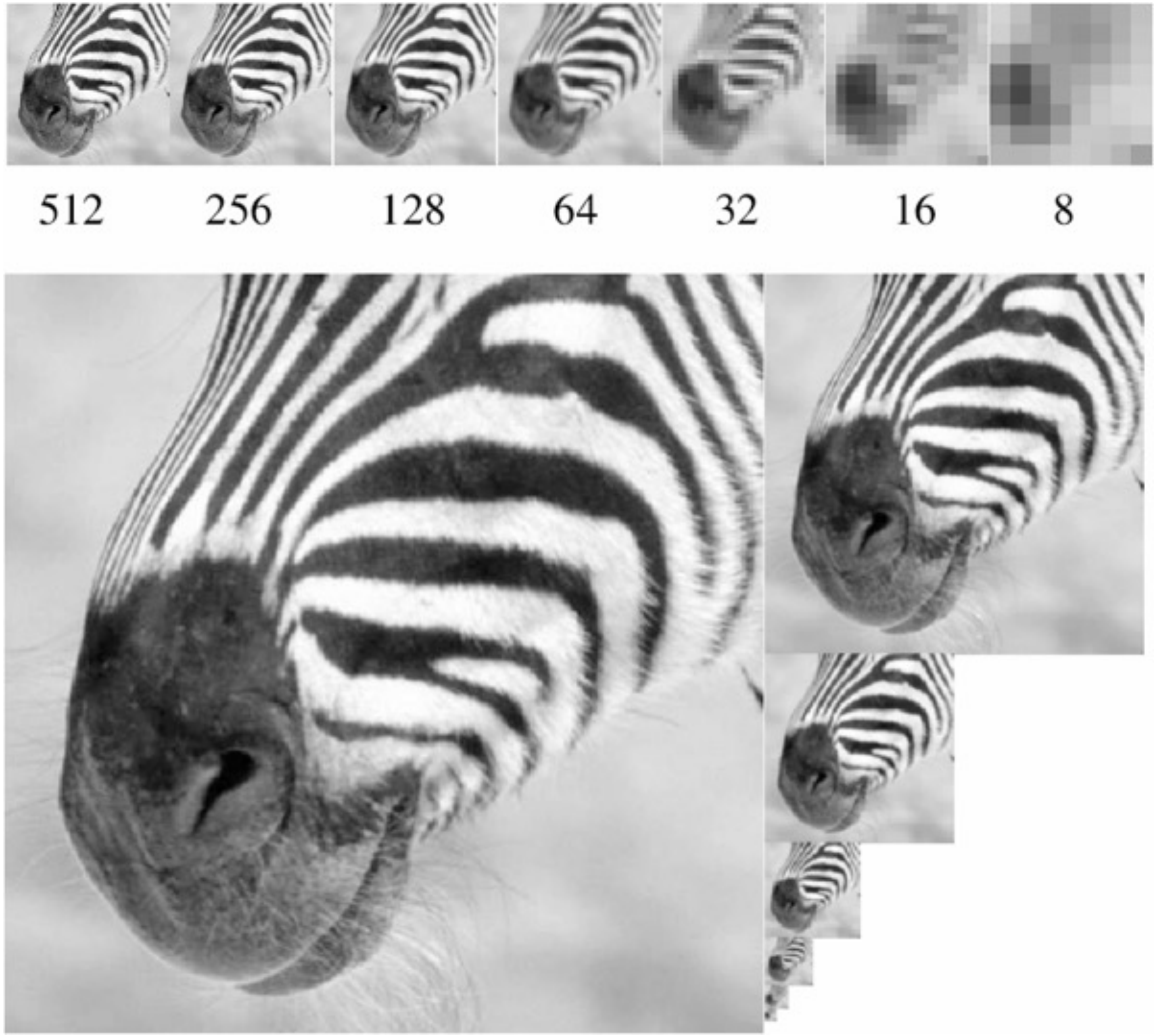
until min resolution reached



Question: How much bigger than the original image is the whole pyramid?

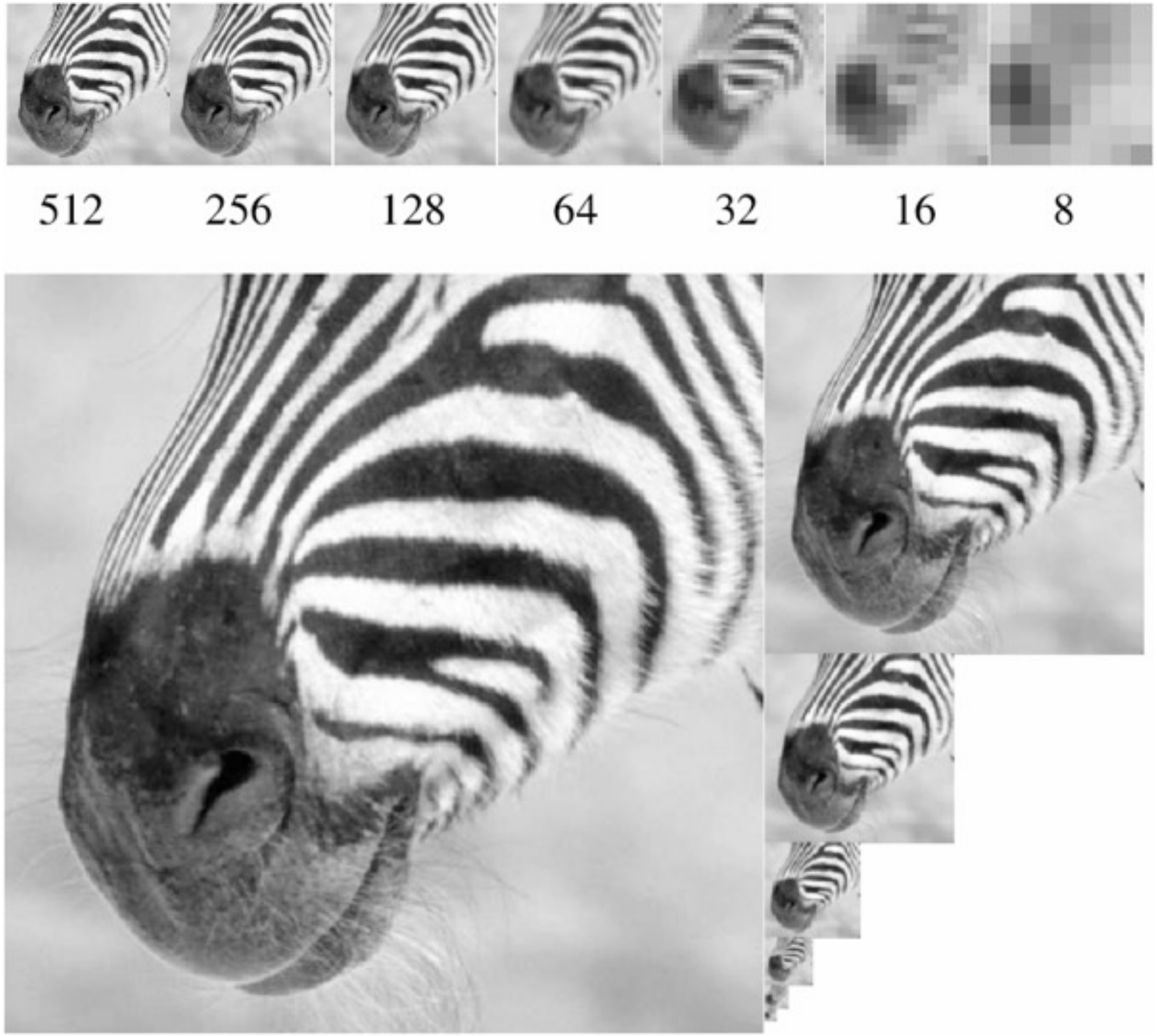
Answer: Just 4/3 times the size of the original image! (How did I come up with this number?)

Some properties of the Gaussian pyramid



What happens to the details of the image?

Some properties of the Gaussian pyramid

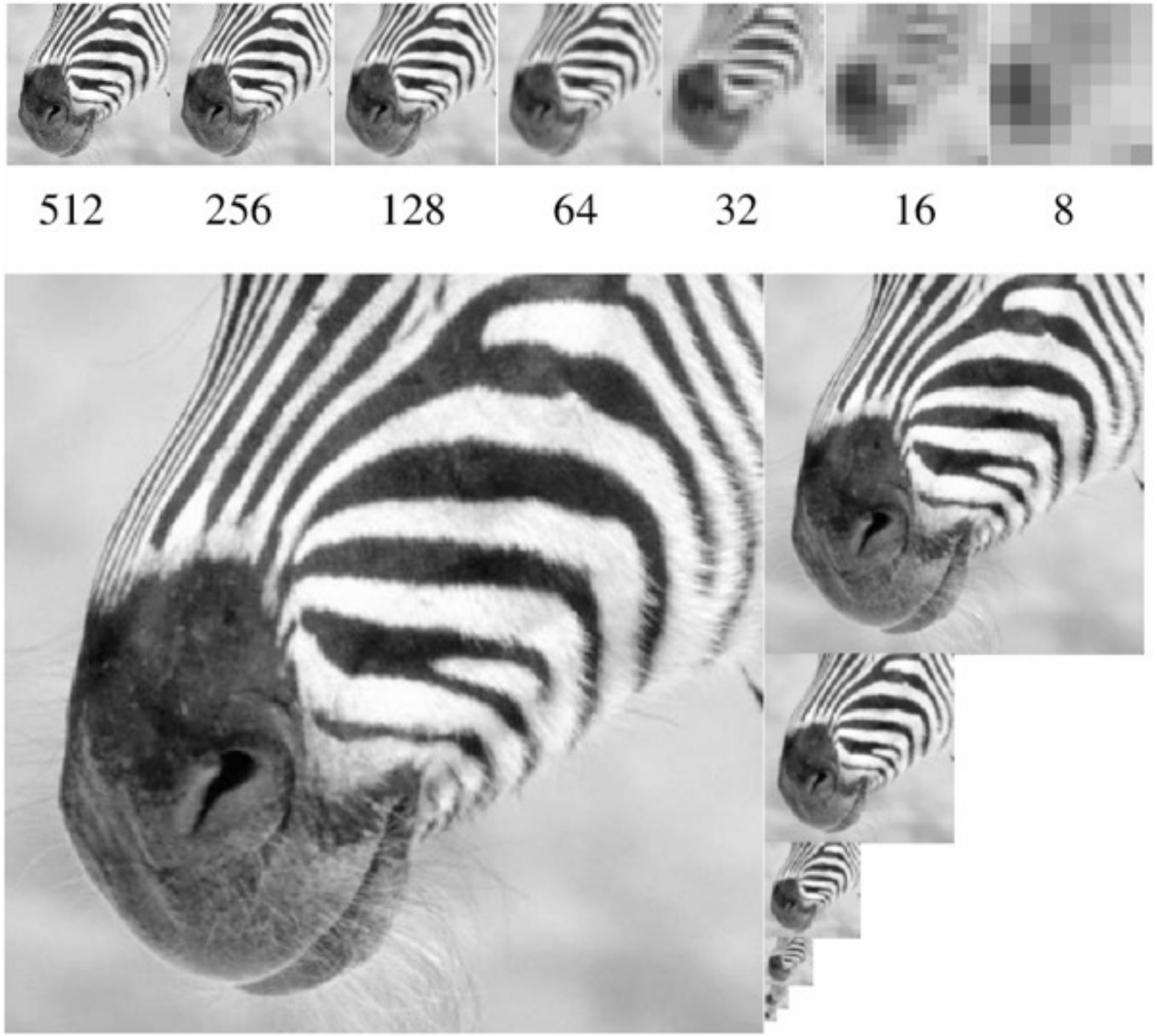


What happens to the details of the image?

- They get smoothed out as we move to higher levels.

What is preserved at the higher levels?

Some properties of the Gaussian pyramid



What happens to the details of the image?

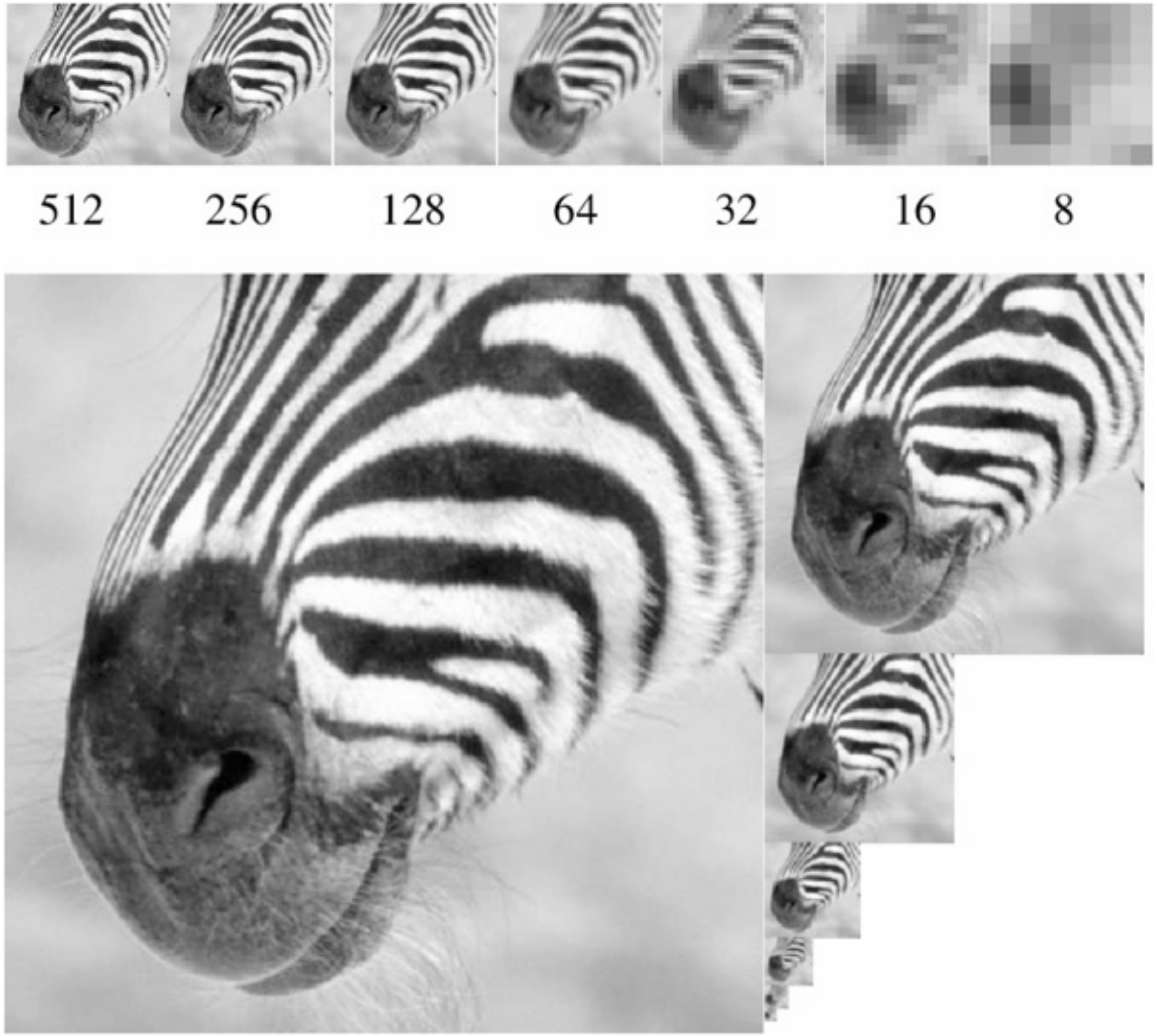
- They get smoothed out as we move to higher levels.

What is preserved at the higher levels?

- Mostly large uniform regions in the original image.

How would you reconstruct the original image from the image at the upper level?

Some properties of the Gaussian pyramid



What happens to the details of the image?

- They get smoothed out as we move to higher levels.

What is preserved at the higher levels?

- Mostly large uniform regions in the original image.

How would you reconstruct the original image from the image at the upper level?

- That’s not possible.

Blurring is lossy

level 0

level 1 (before downsampling)

What does the residual look like?

Blurring is lossy

-

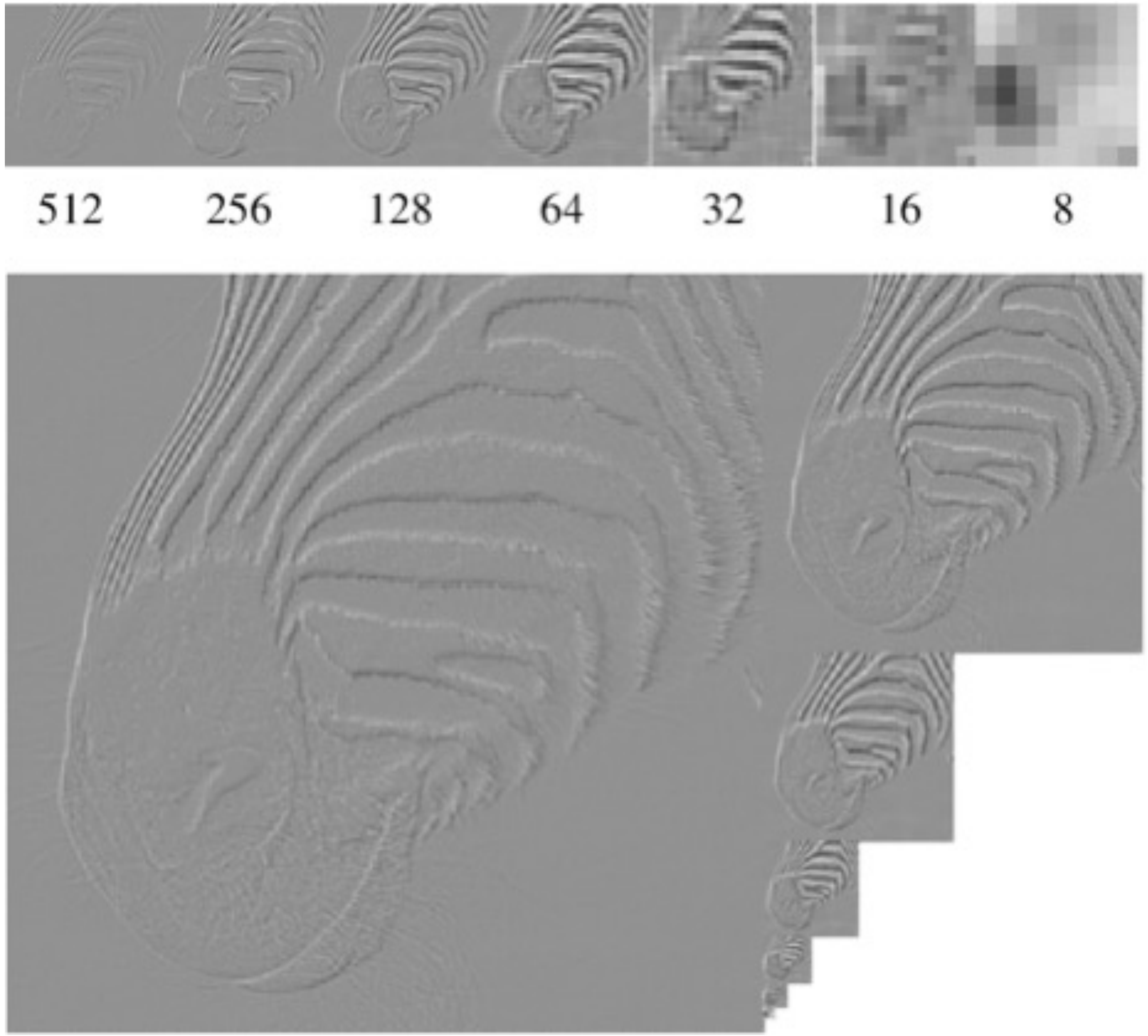
=

level 0 | level 1 (before downsampling) | residual

Can we make a pyramid that is lossless?

Laplacian image pyramid

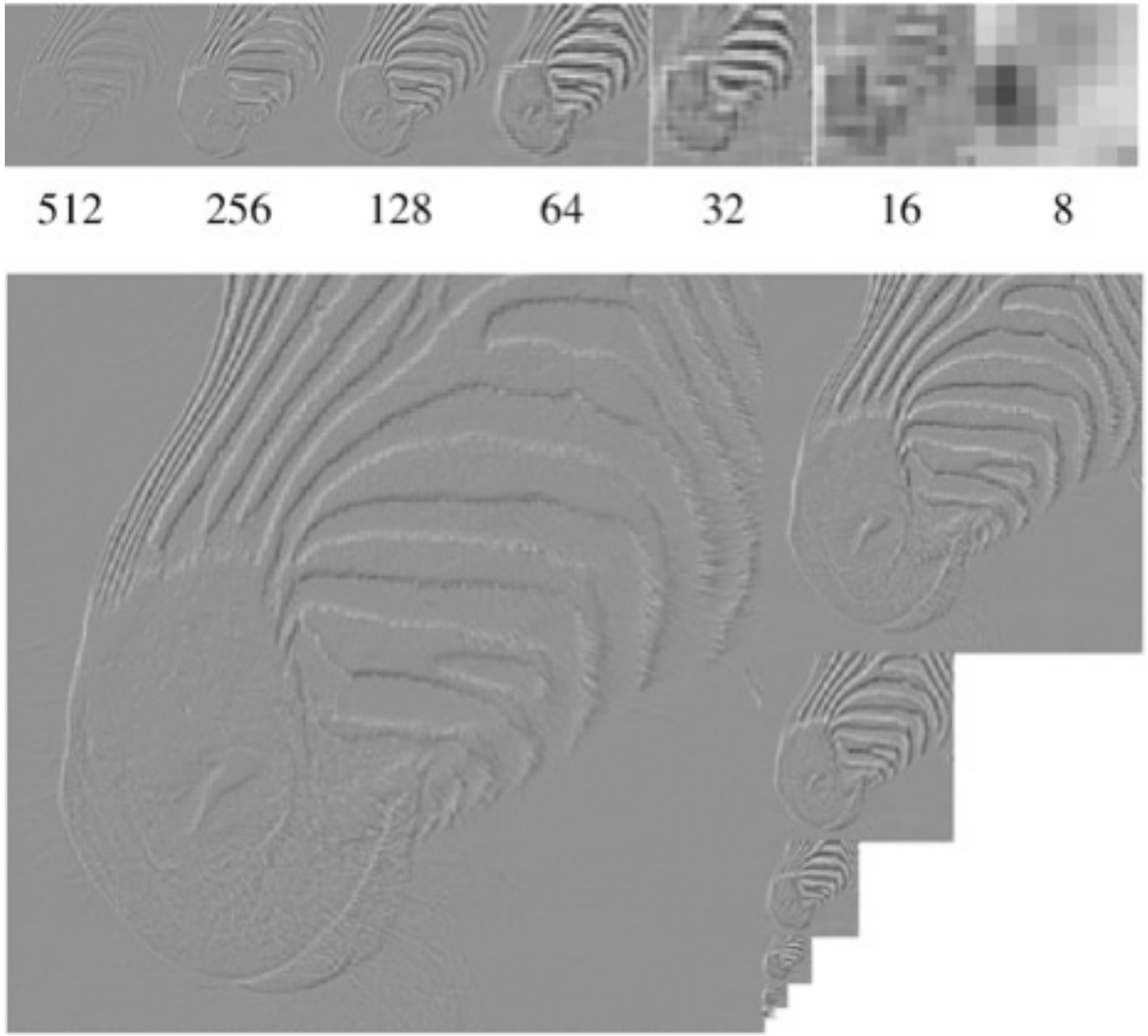
Laplacian image pyramid



At each level, retain the residuals instead of the blurred images themselves.

Can we reconstruct the original image using the pyramid?

Laplacian image pyramid



At each level, retain the residuals instead of the blurred images themselves.

Can we reconstruct the original image using the pyramid?

- Yes we can!

What do we need to store to be able to reconstruct the original image?

Let's start by looking at just one level

level 0

=

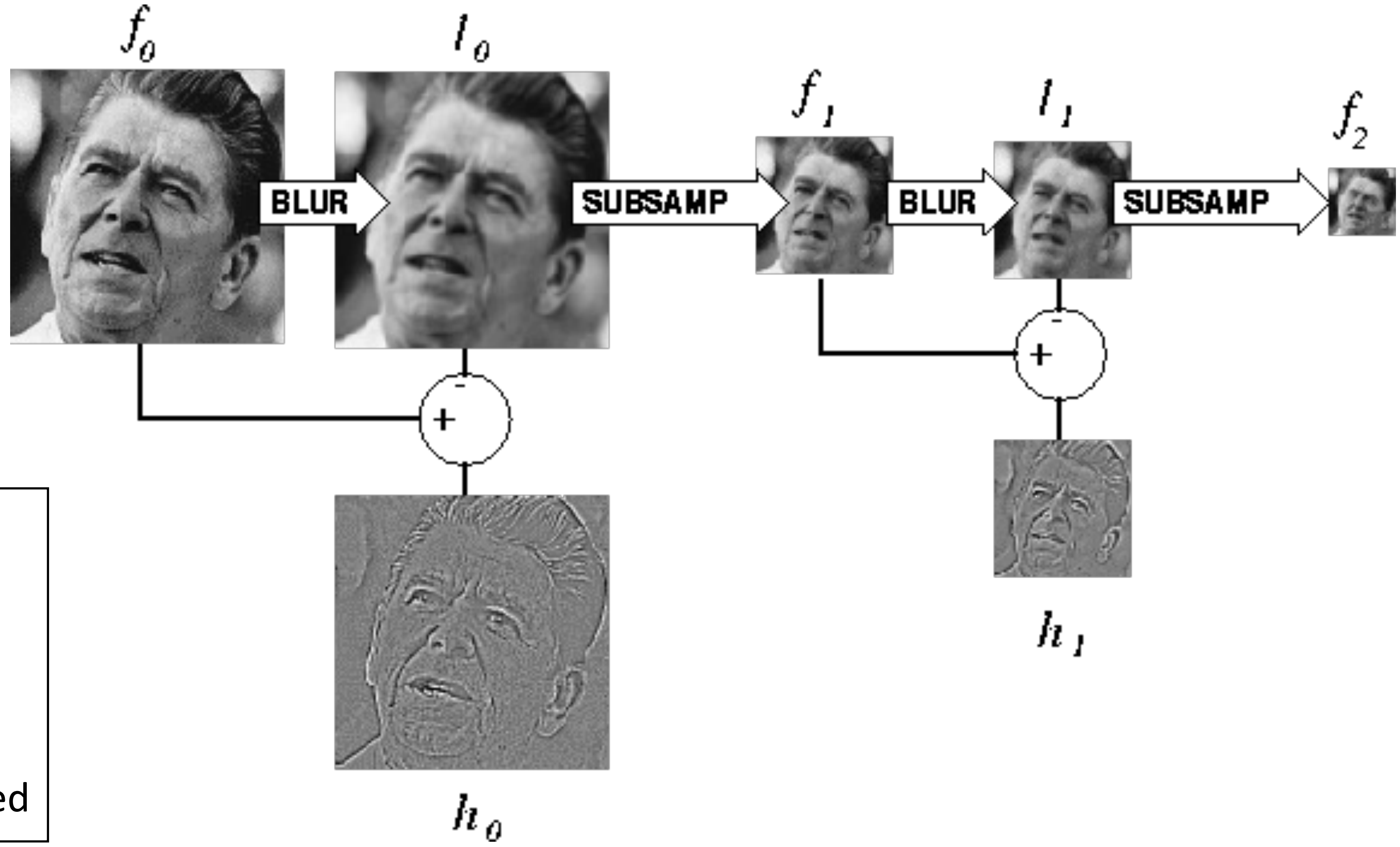
level 1 (upsampled)

+

residual

Does this mean we need to store both residuals and the blurred copies of the original?

Constructing a Laplacian pyramid



Algorithm

repeat:

filter

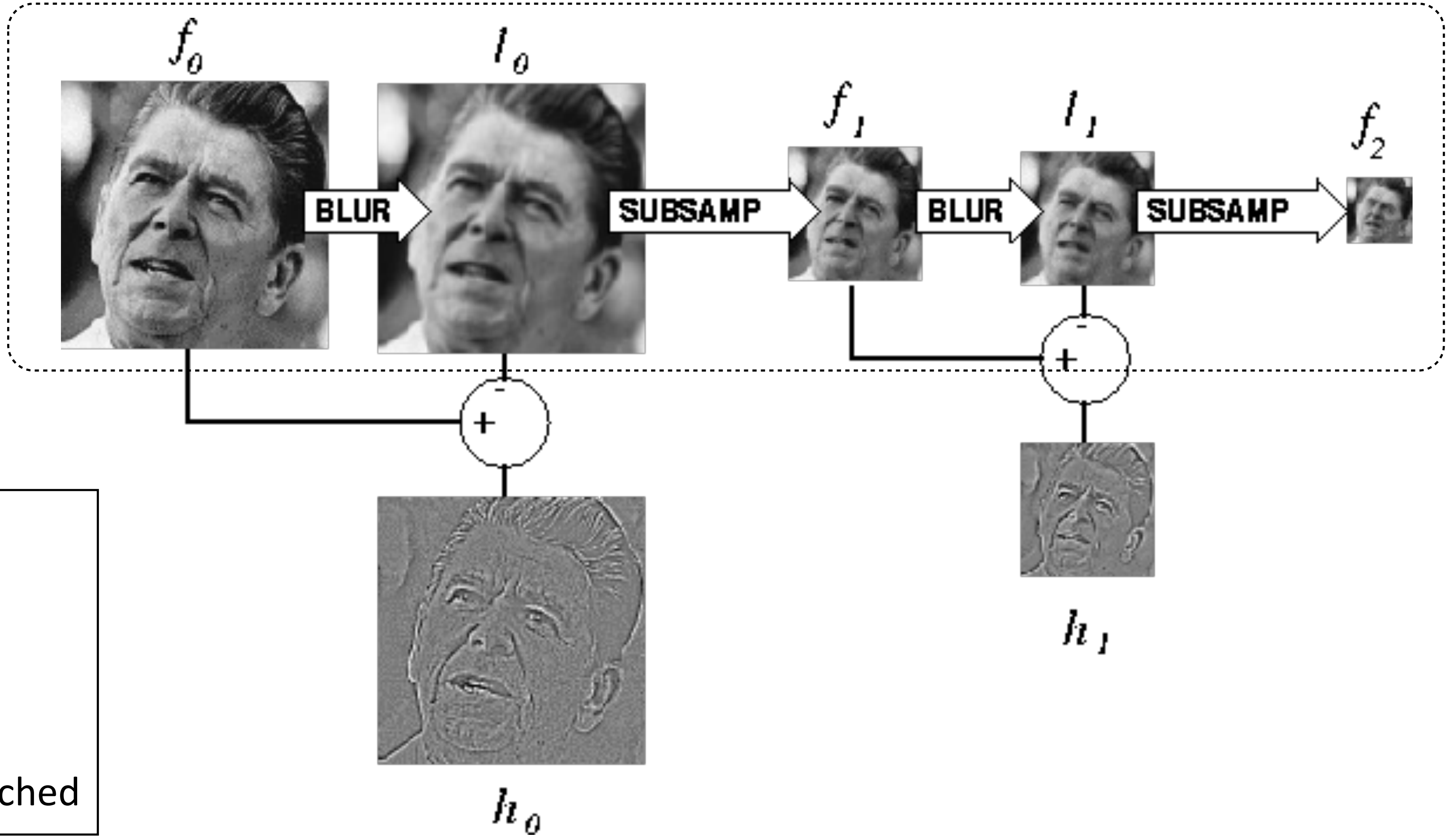
compute residual

subsample

until min resolution reached

Constructing a Laplacian pyramid

What is this part?



Algorithm

repeat:

filter

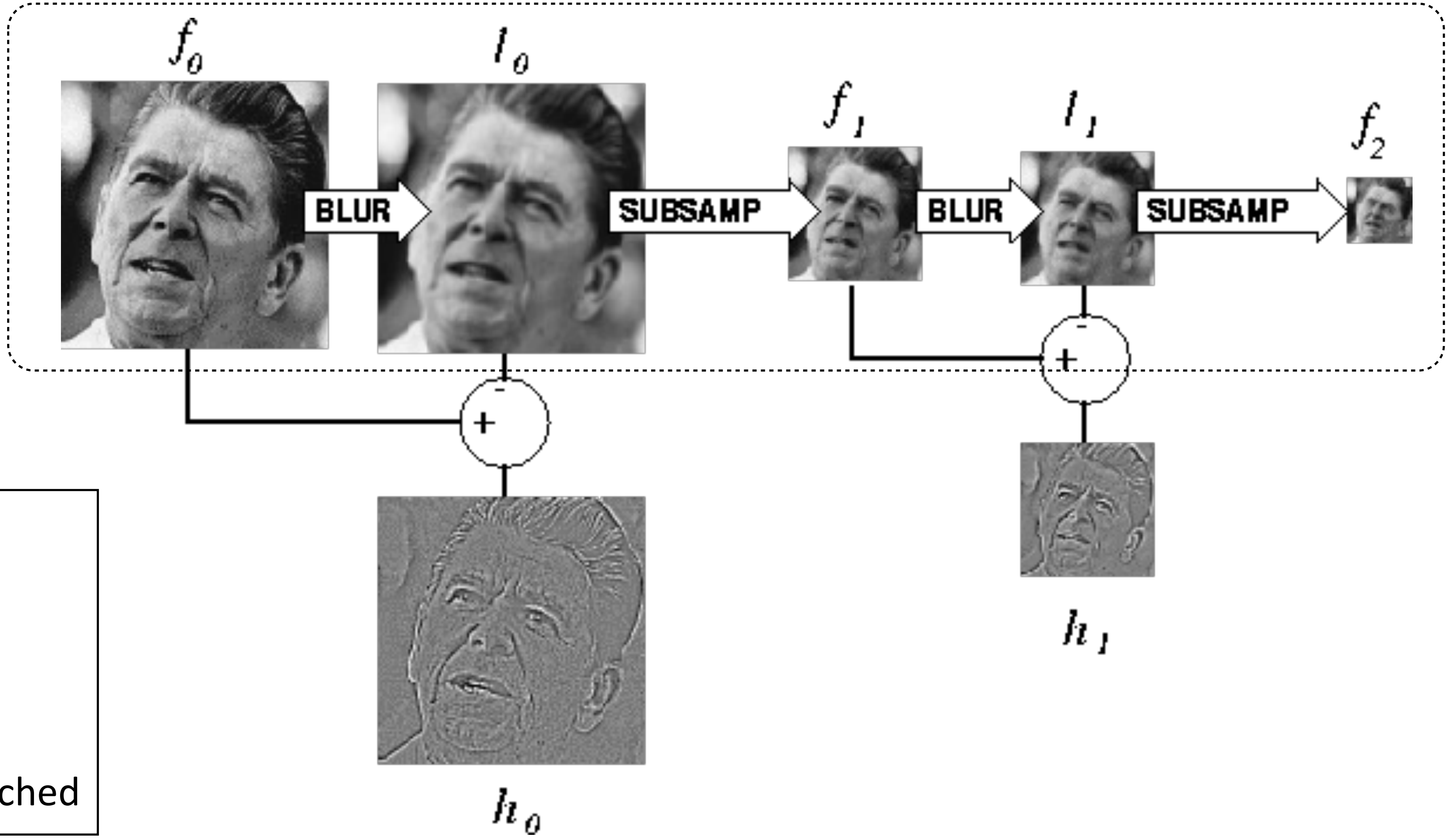
compute residual

subsample

until min resolution reached

Constructing a Laplacian pyramid

It's a Gaussian pyramid.



Algorithm

repeat:

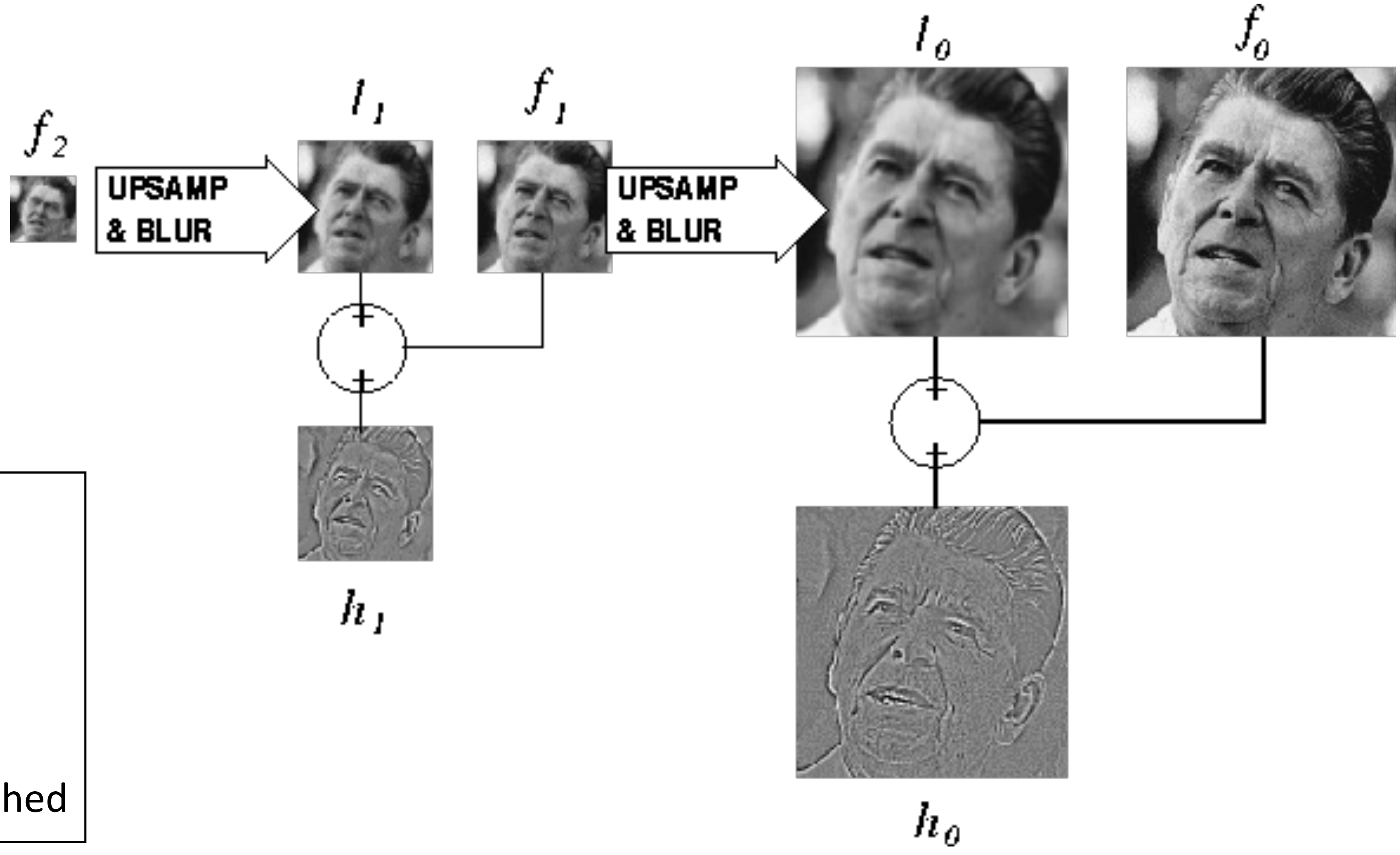
filter

compute residual

subsample

until min resolution reached

Reconstructing the original image



Algorithm

repeat:

upsample

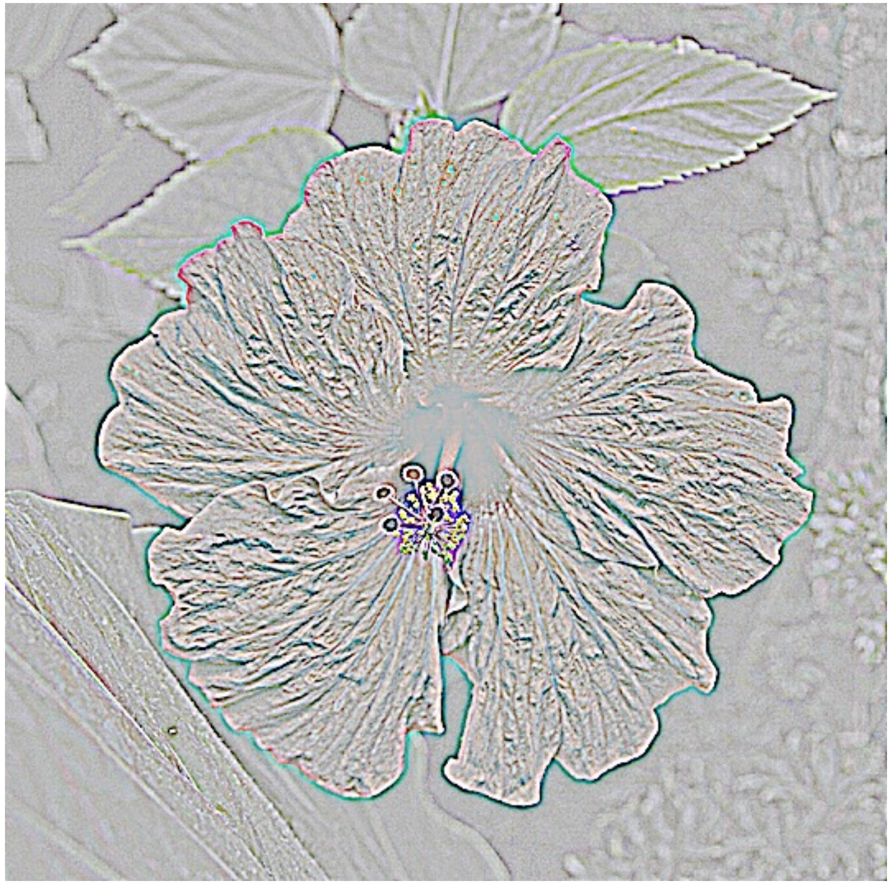
sum with residual

until orig resolution reached

Gaussian vs Laplacian Pyramid




Shown in opposite order for space.




Which one takes more space to store?


Why is it called a Laplacian pyramid?

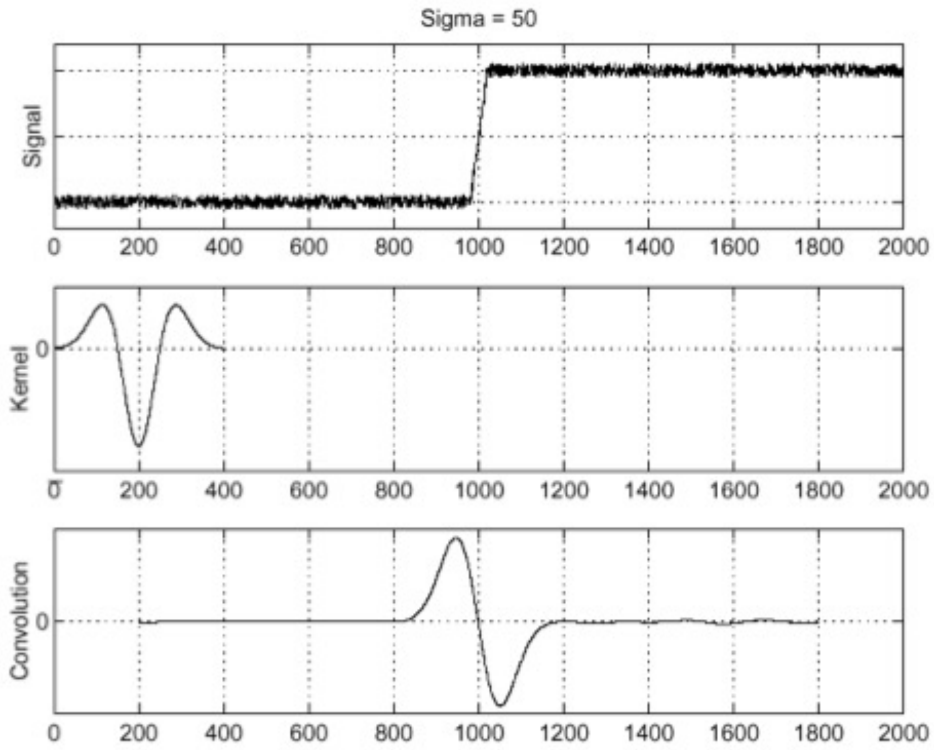
Reminder: Laplacian of Gaussian (LoG) filter

As with derivative, we can combine Laplace filtering with Gaussian filtering

input

Laplacian of Gaussian

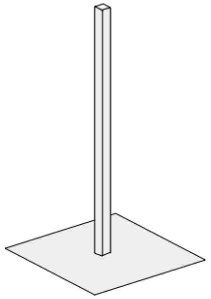
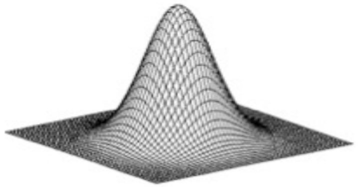
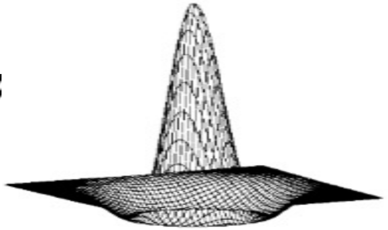
output



"zero crossings" at edges

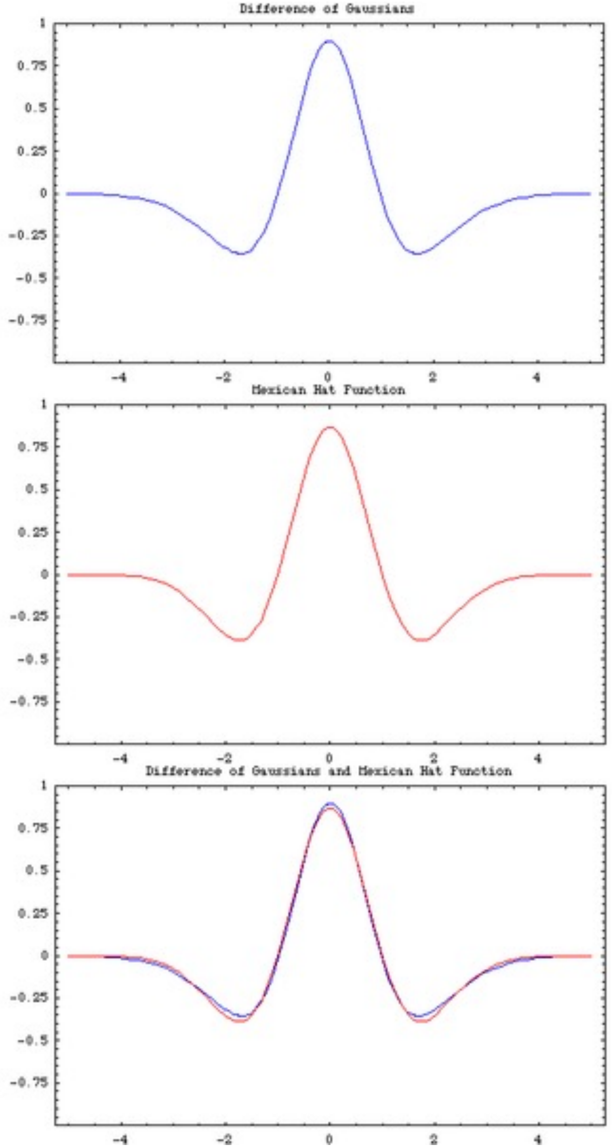
Why is it called a Laplacian pyramid?

 -  = 

 -  ≈ 

unit | Gaussian | Laplacian

Difference of Gaussians approximates the Laplacian



Still used extensively

Still used extensively

foreground details enhanced, background details reduced

input image

user-provided mask

Other types of pyramids

Steerable pyramid: At each level keep multiple versions, one for each direction.

Wavelets: Huge area in image processing (see 18-793).

What are image pyramids used for?

image compression

multi-scale texture mapping

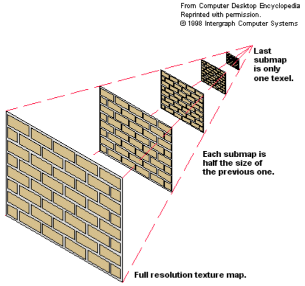
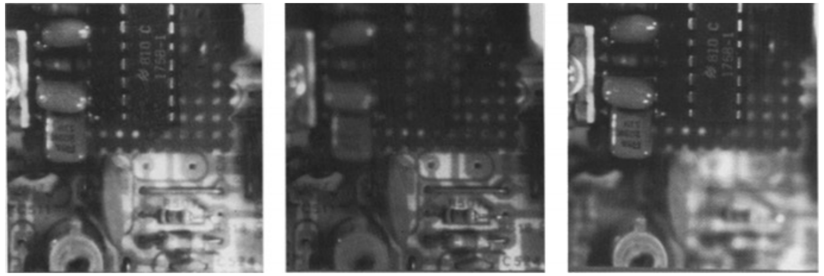
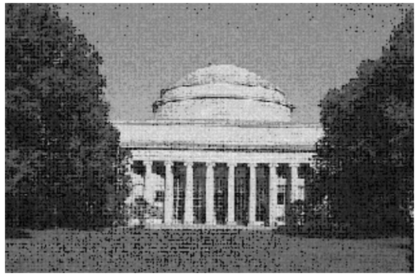


image blending

focal stack compositing

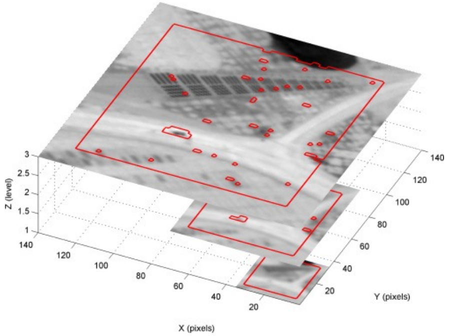


denoising



multi-scale detection

multi-scale registration



Fourier series

Basic building block

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get any *periodic* signal you want!

Basic building block

$$A \sin(\omega x + \phi)$$

amplitude

sinusoid

angular frequency

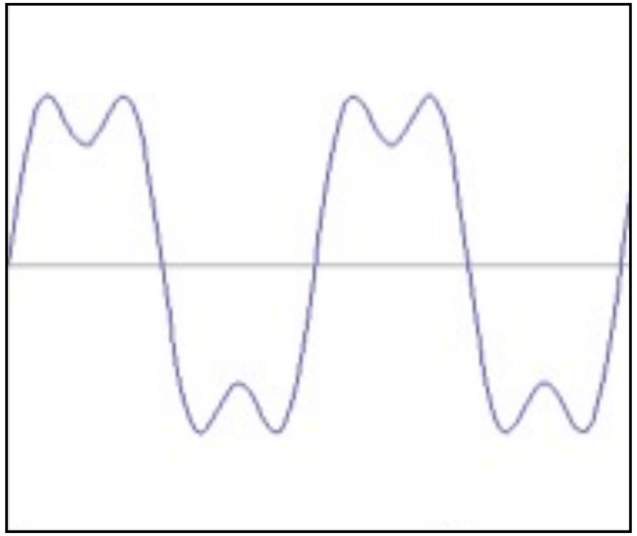
variable

phase

Fourier’s claim: Add enough of these to get <u>any</u> *periodic* signal you want!

Examples

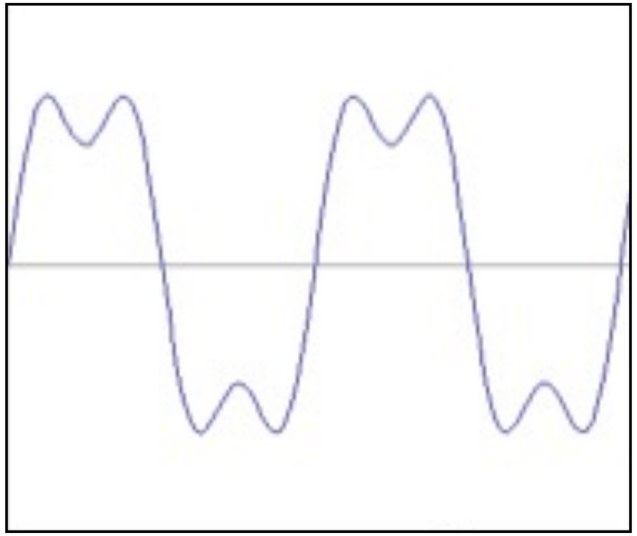
How would you generate this function?



= ? + ?

Examples

How would you generate this function?



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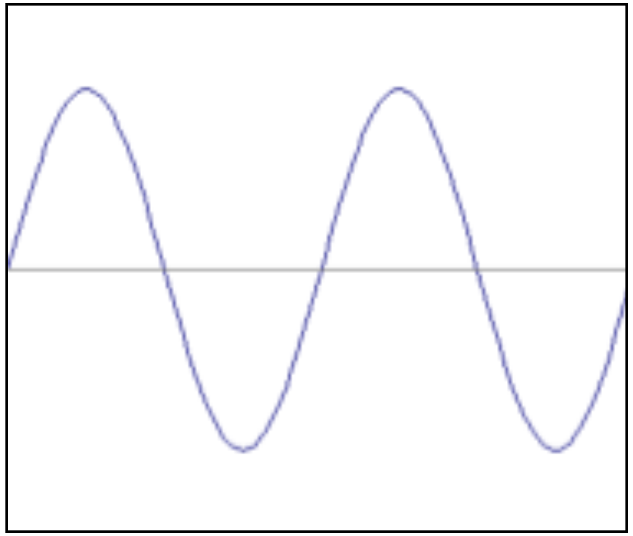
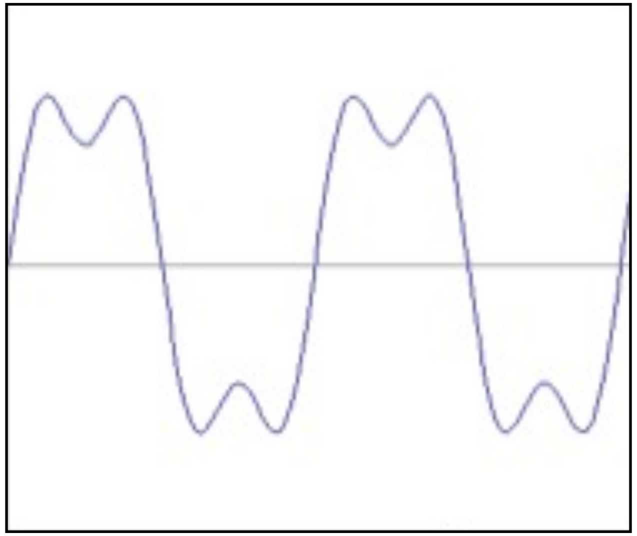


$\sin(2\pi x)$

\+ ?

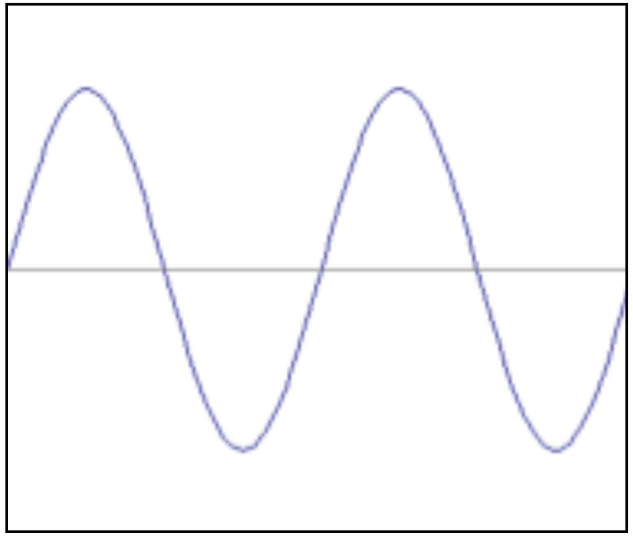
Examples

How would you generate this function?



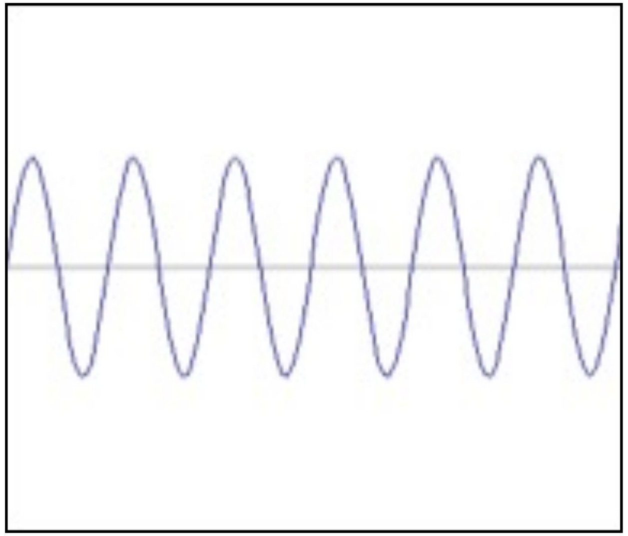
$$f(x) = \sin(2\pi x) + \frac{1}{3}\sin(2\pi 3x)$$

=



$$\sin(2\pi x)$$

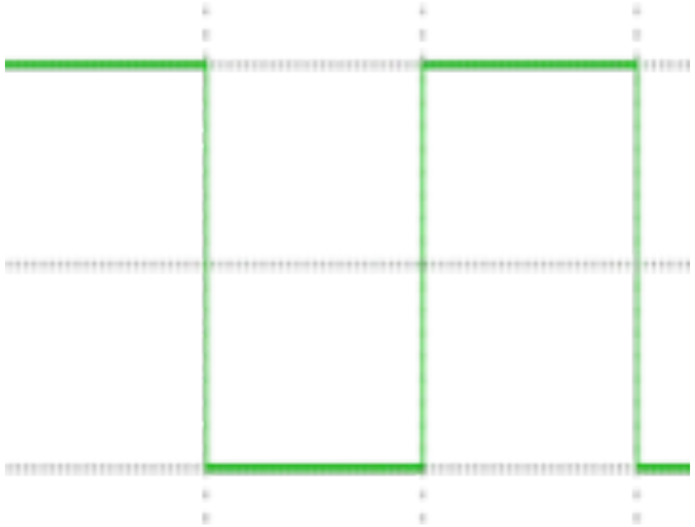
+



$$\frac{1}{3}\sin(2\pi 3x)$$

Examples

How would you generate this function?

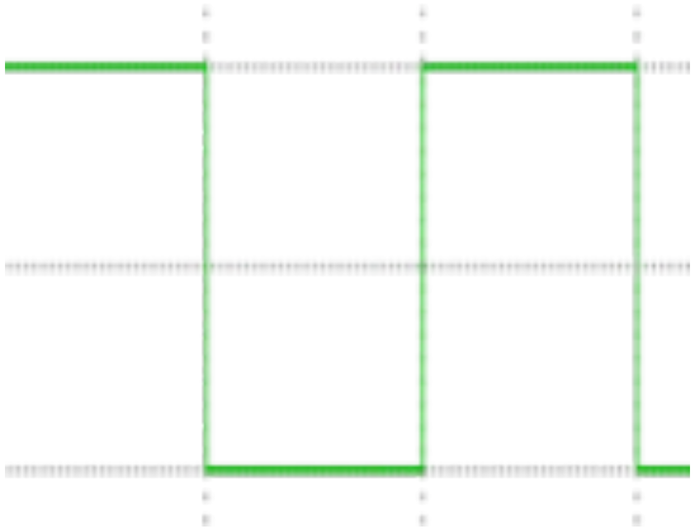


square wave

= ? + ?

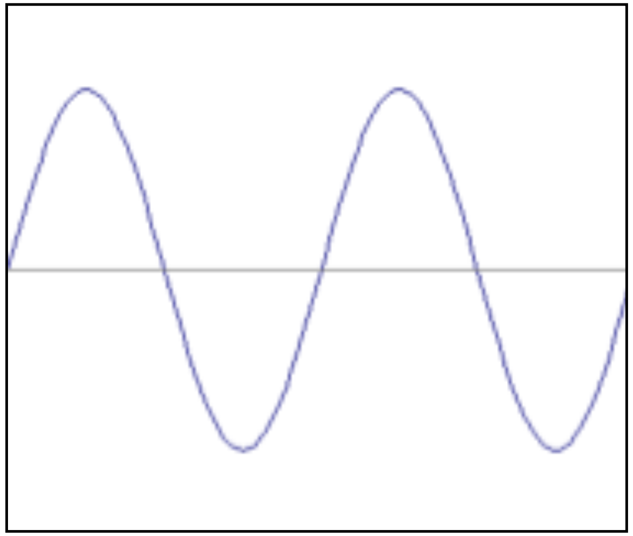
Examples

How would you generate this function?

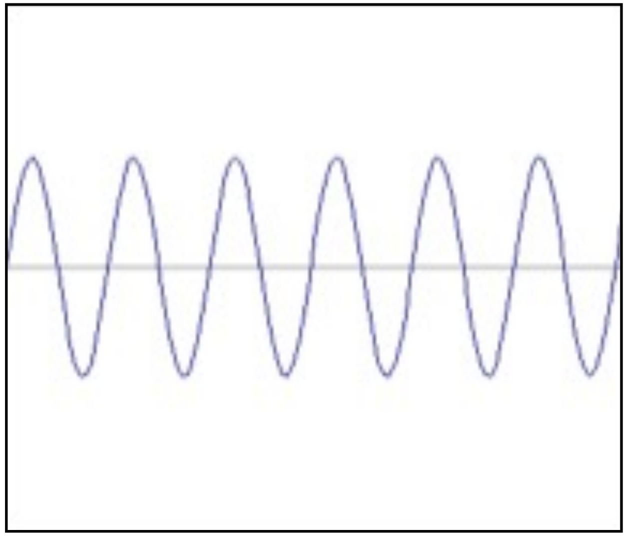


square wave

≈



+



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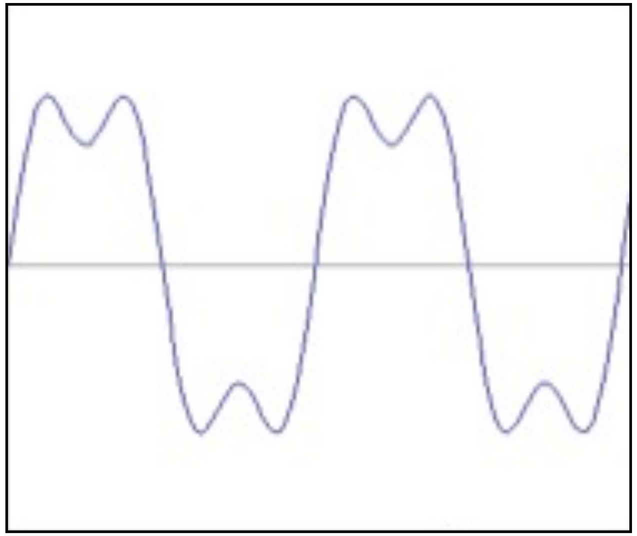
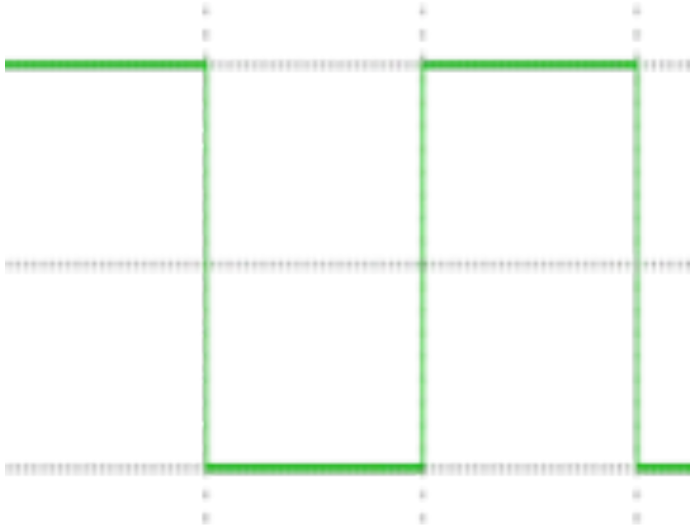


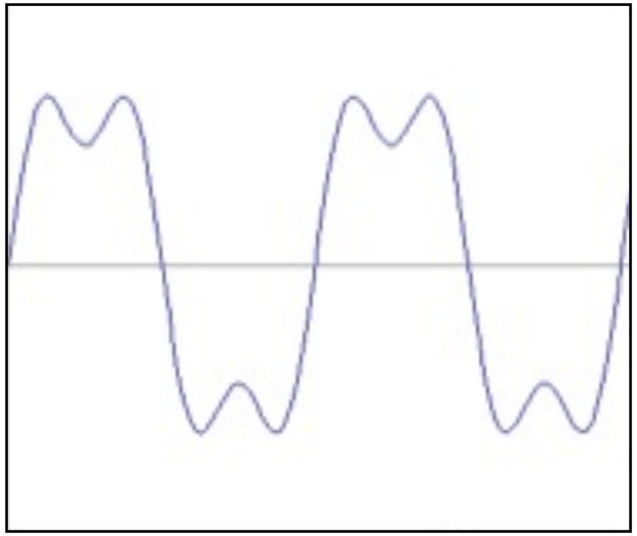
Examples

How would you generate this function?

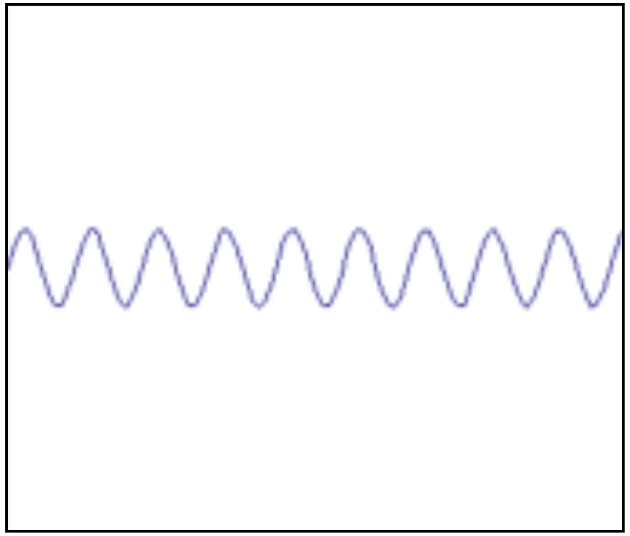


square wave

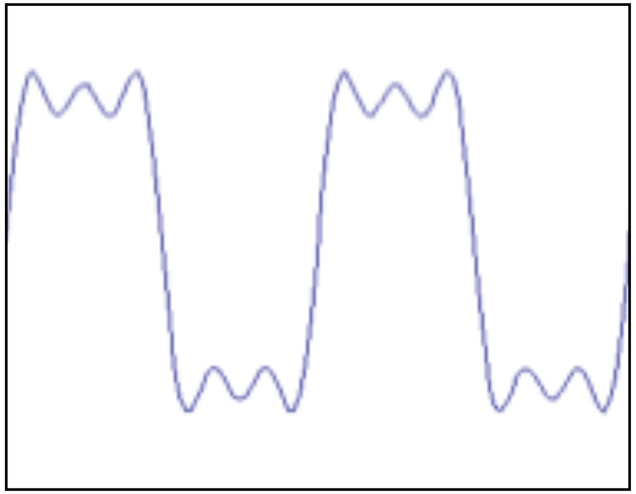
≈



+

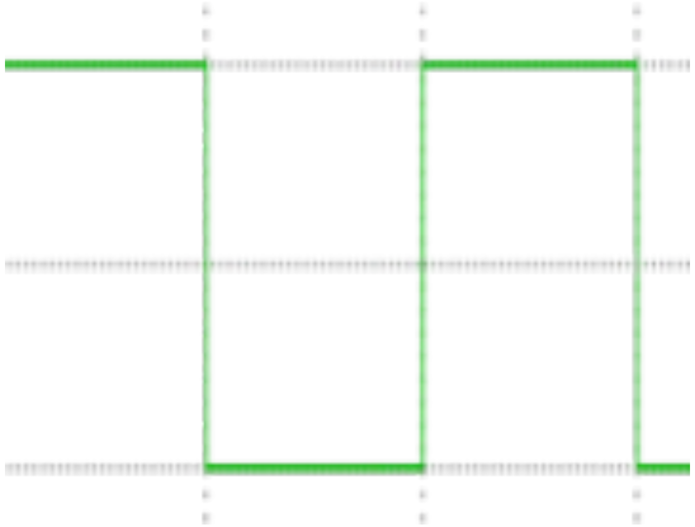


=



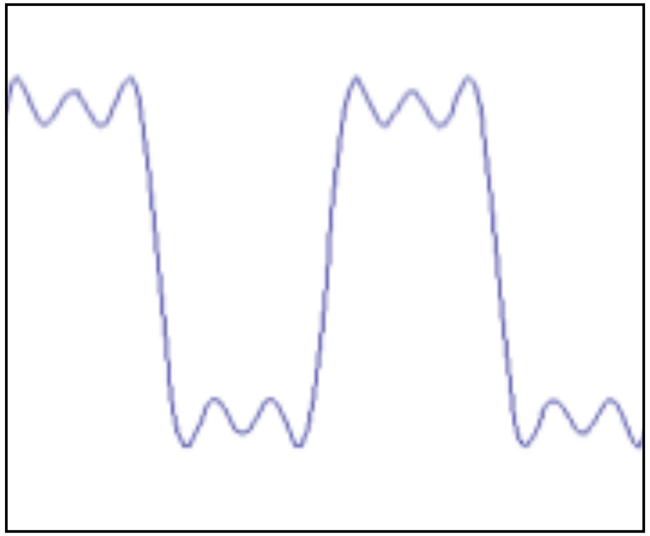
Examples

How would you generate this function?



square wave

≈



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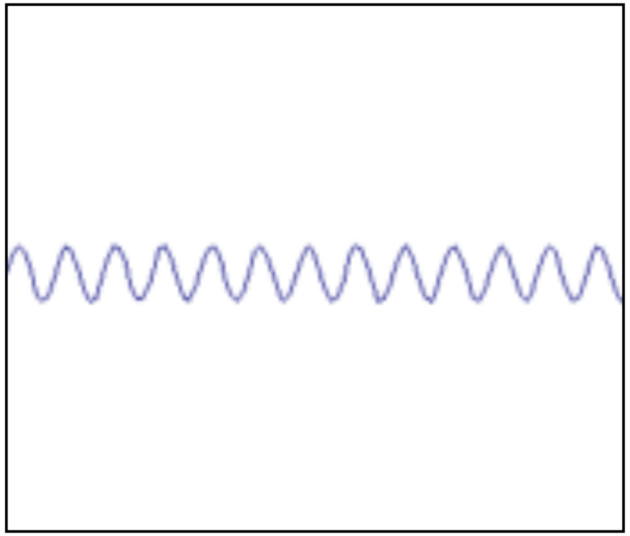

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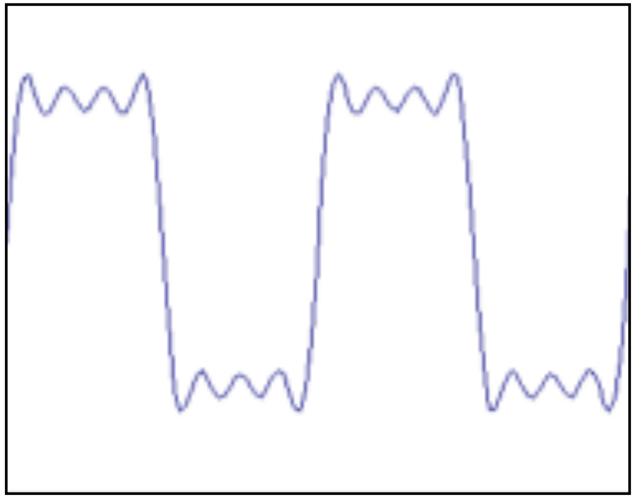
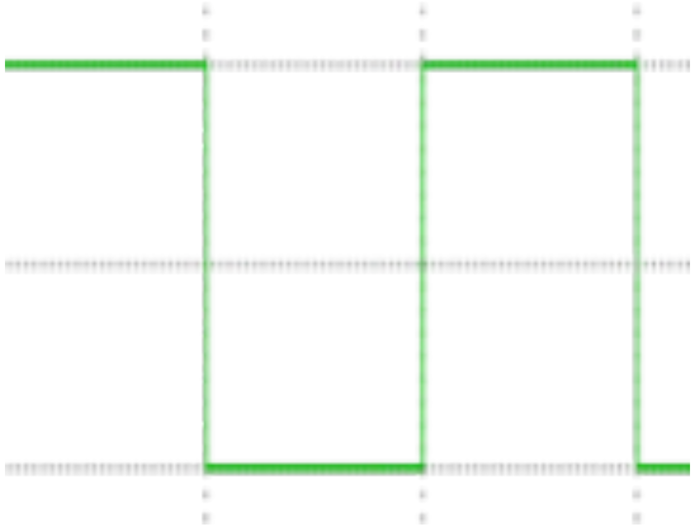


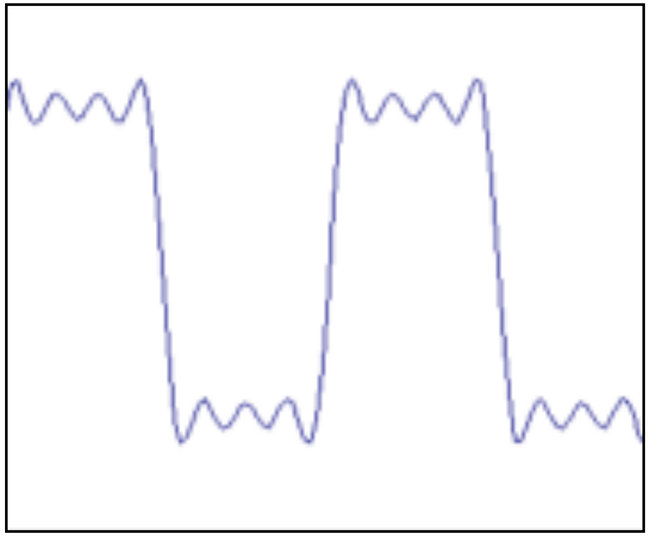
Examples

How would you generate this function?



square wave

≈



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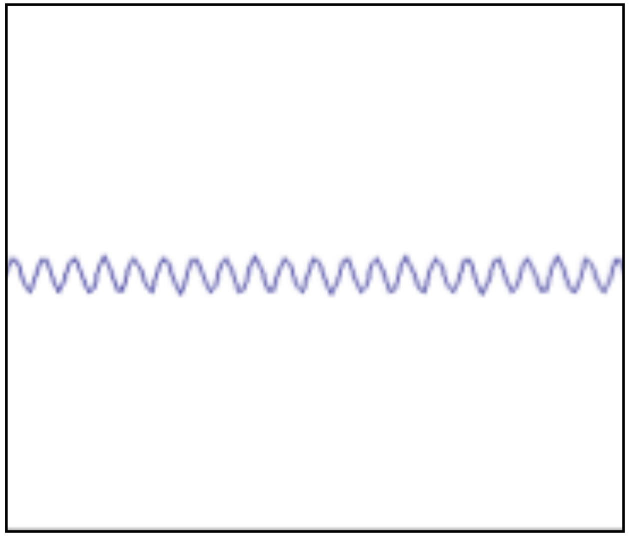

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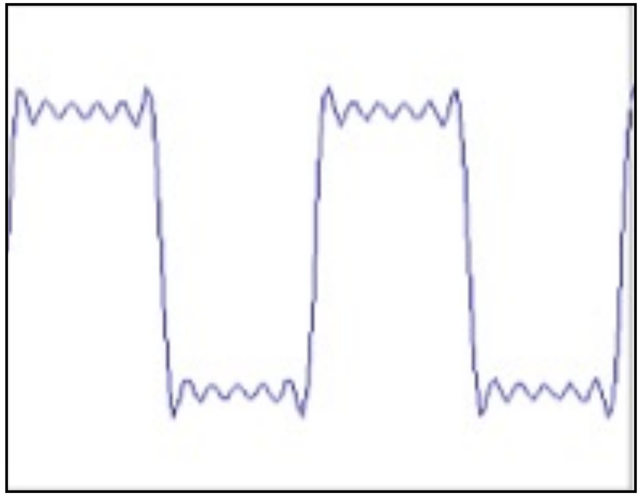
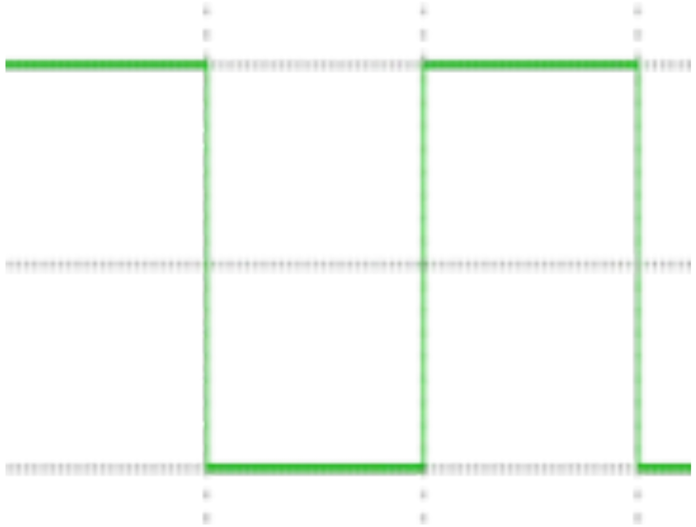


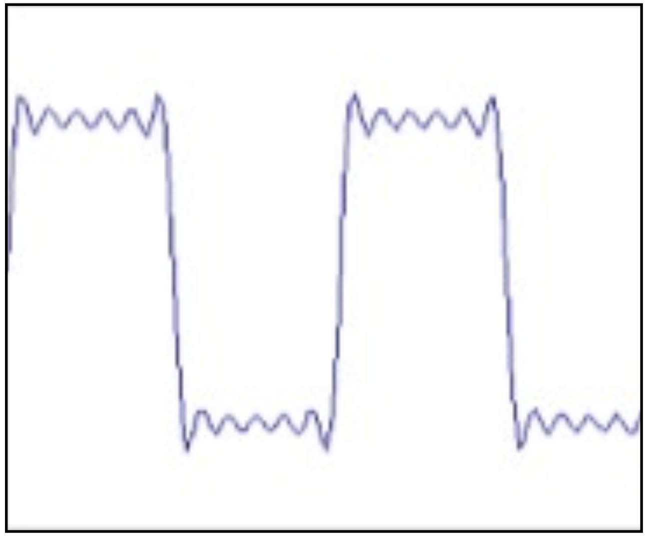
Examples

How would you generate this function?



square wave

≈



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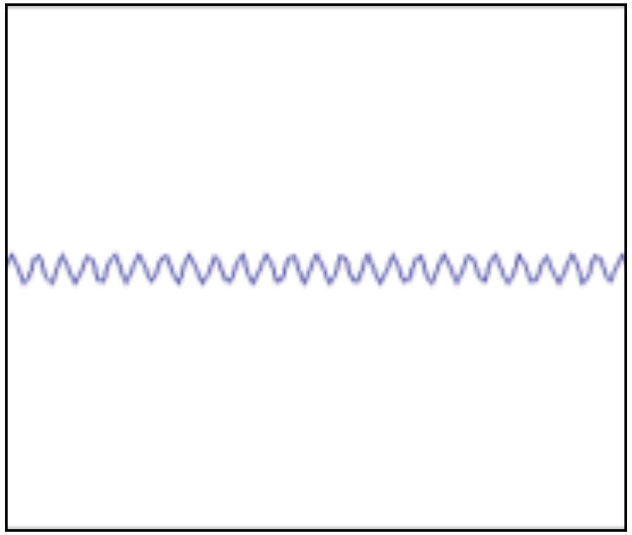

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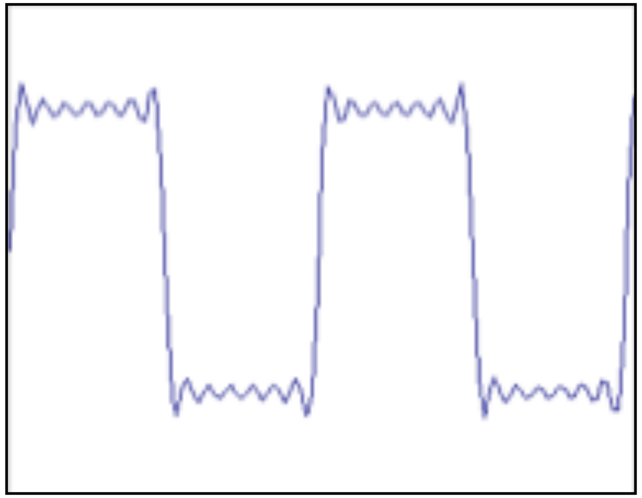
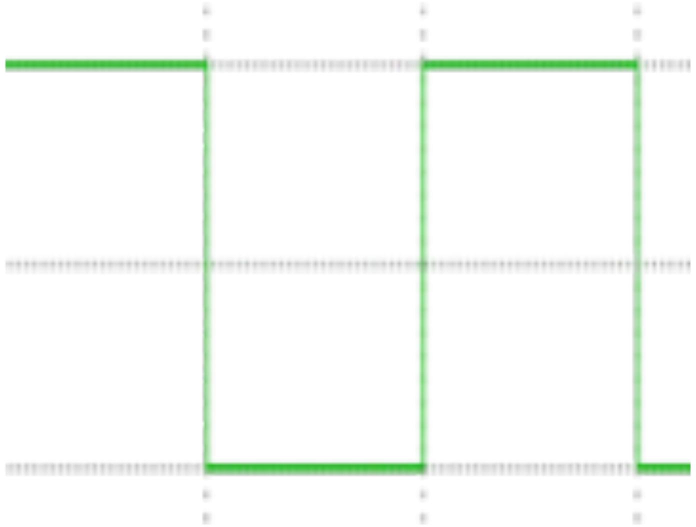


How would you express this mathematically?

Examples



square wave

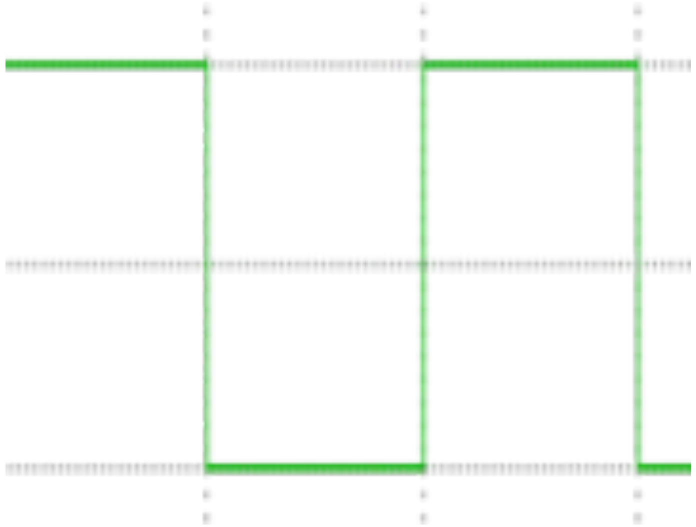
=

$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k x)$$

infinite sum of sine waves

How would could you visualize this in the frequency domain?

Examples

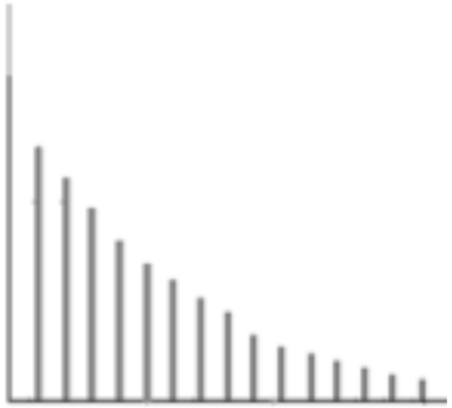


square wave

$$= A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kx)$$

infinite sum of sine waves

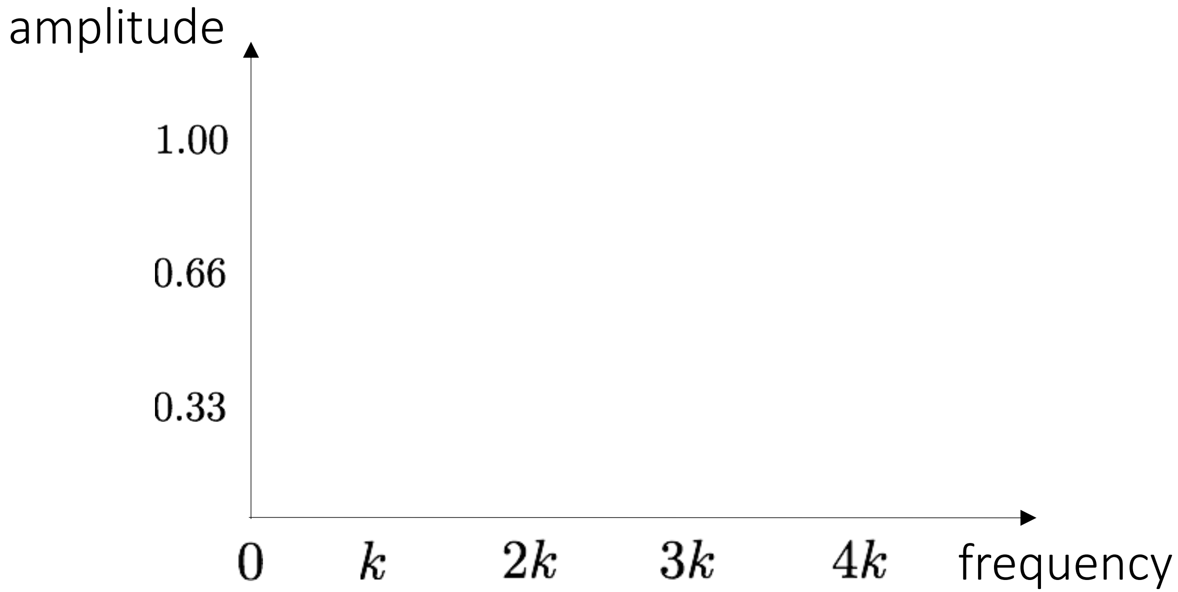
magnitude



frequency

Frequency domain

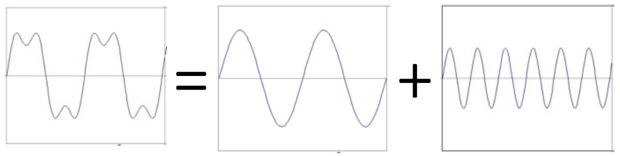
Visualizing the frequency spectrum

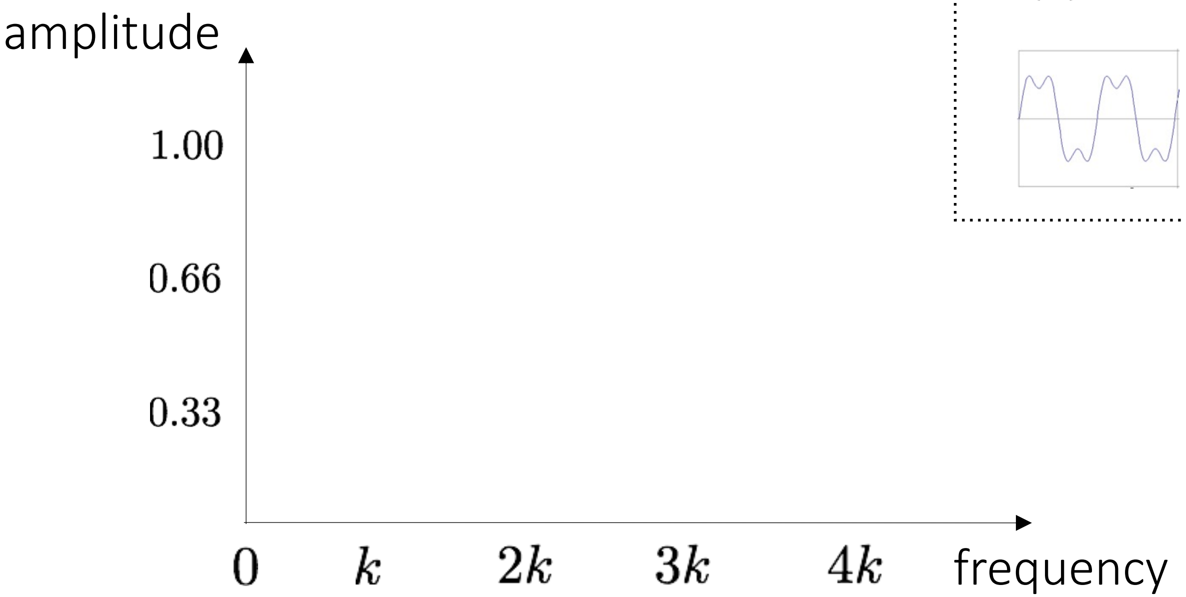


Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$





Visualizing the frequency spectrum

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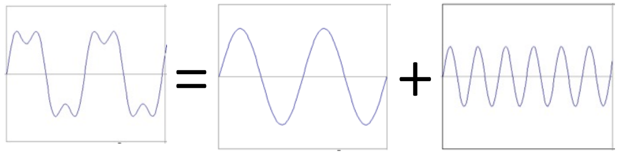
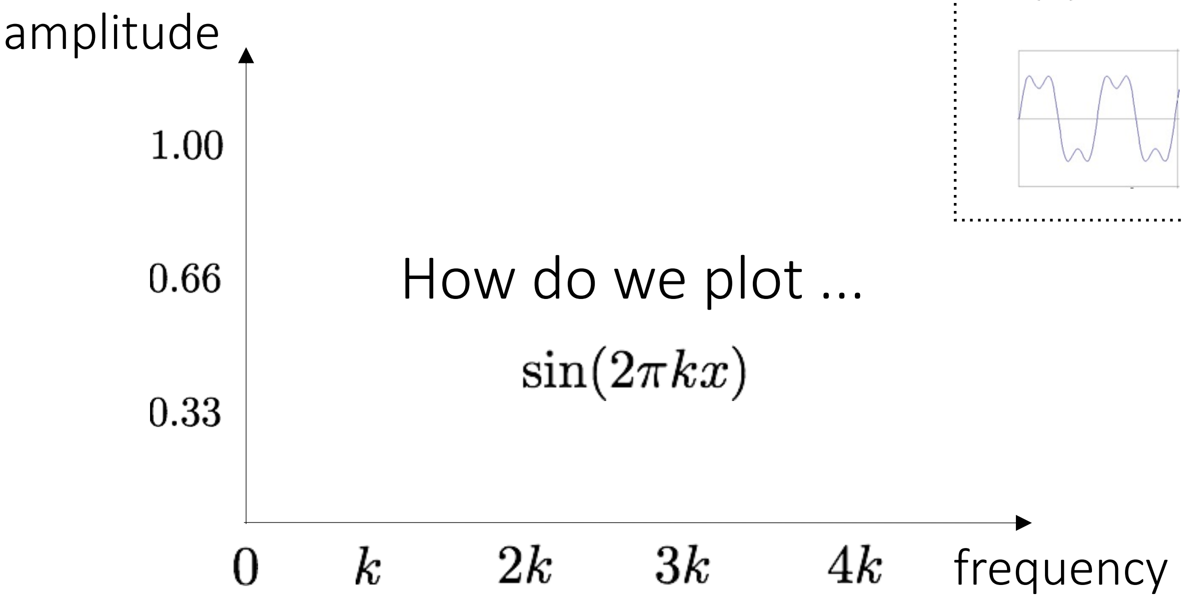




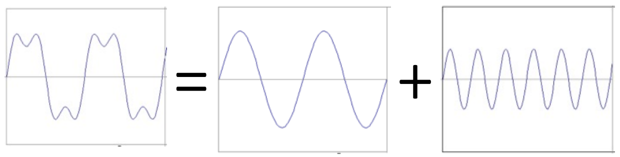
How do we plot ...

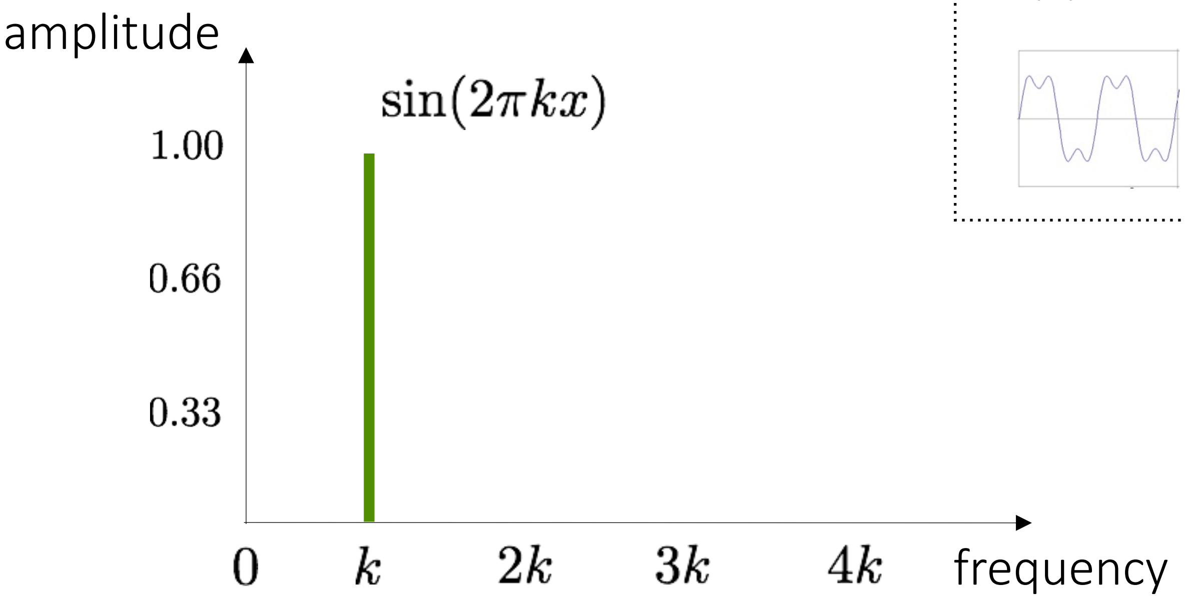
$$\sin(2\pi kx)$$

Visualizing the frequency spectrum

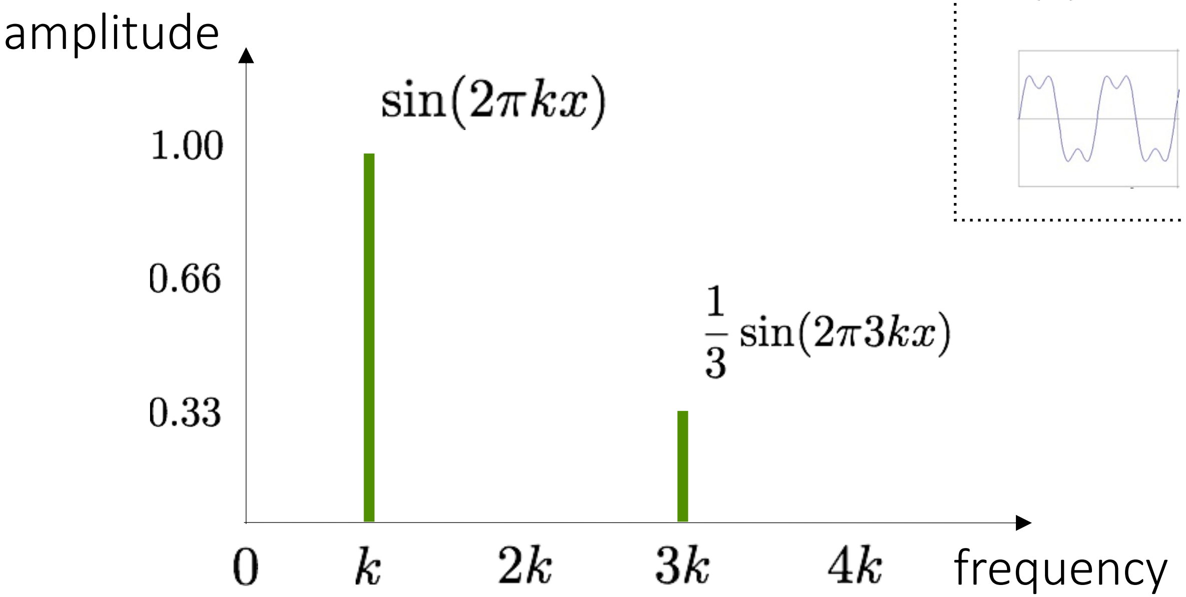
Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$





Visualizing the frequency spectrum



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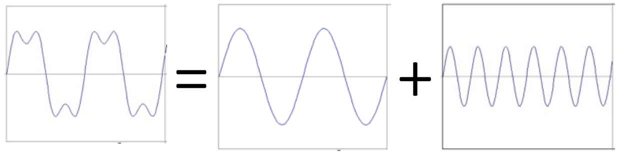

Visualizing the frequency spectrum

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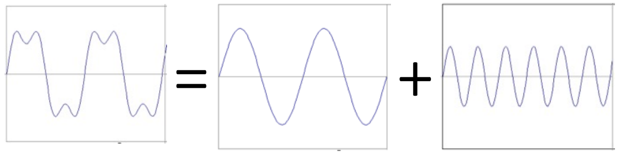
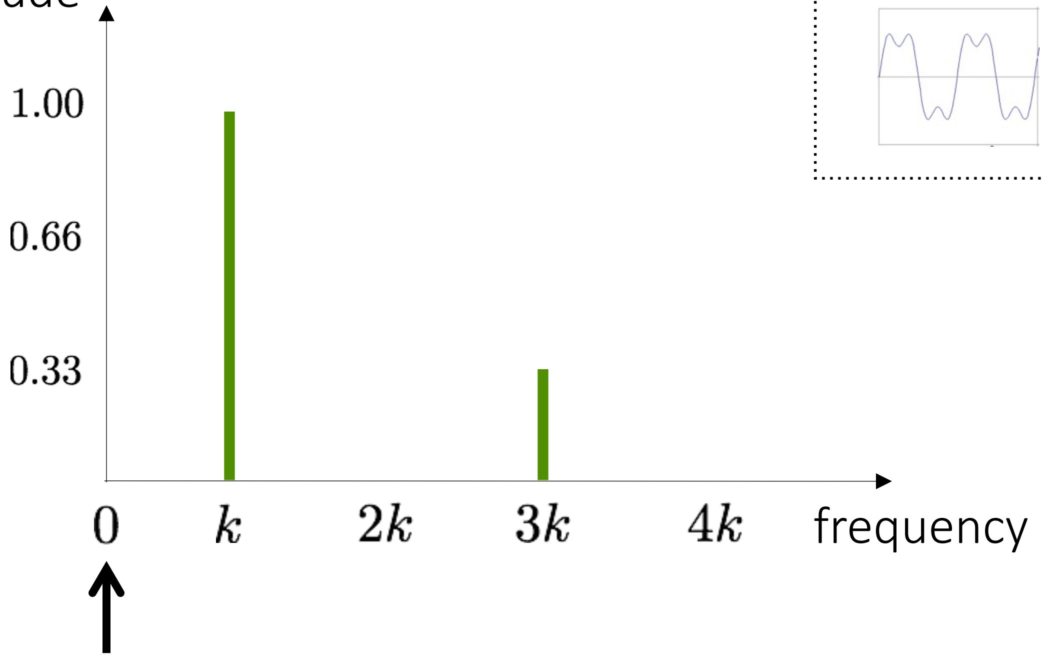
$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$





not visualizing the symmetric negative part

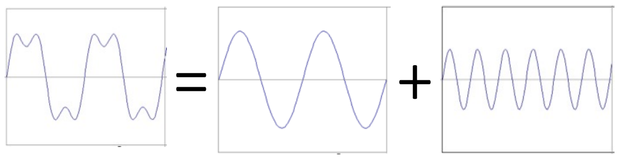
What is at zero frequency?

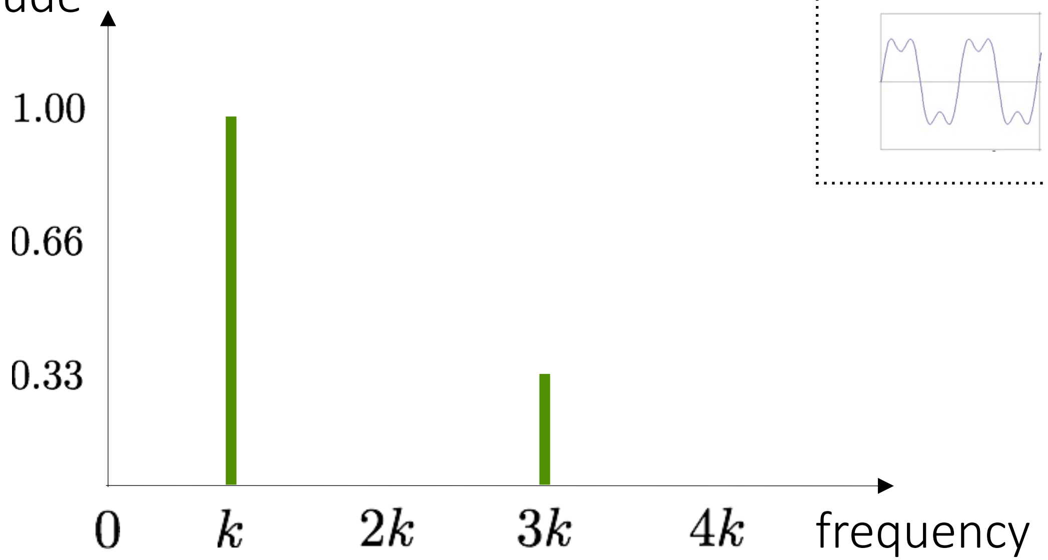
Need to understand this to understand the 2D version!

Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$





not visualizing the symmetric negative part

signal average (zero for a sine wave with no offset)

Need to understand this to understand the 2D version!

Fourier transform

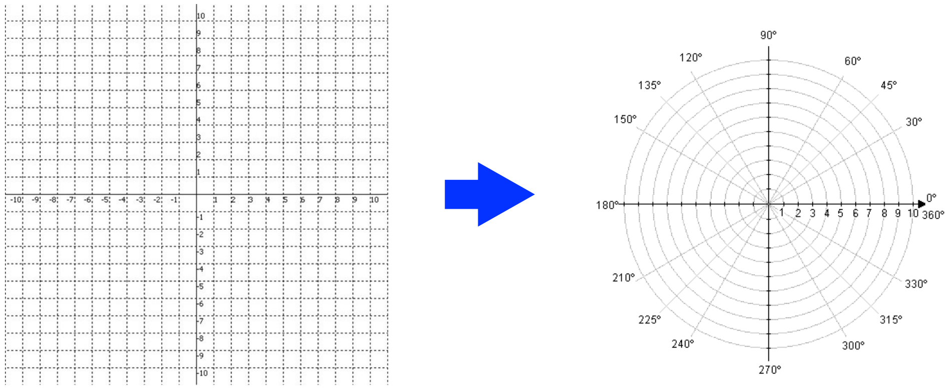
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

what‘s this? what‘s this?



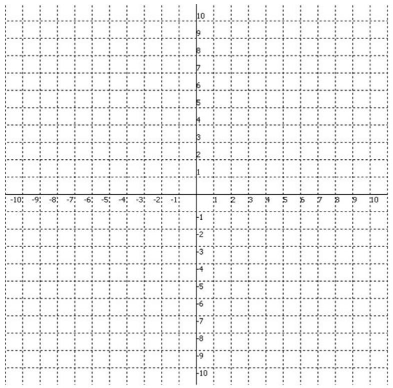
Recalling some basics

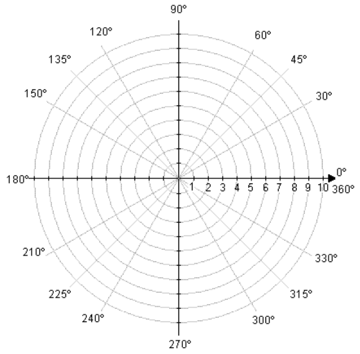
Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary





Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

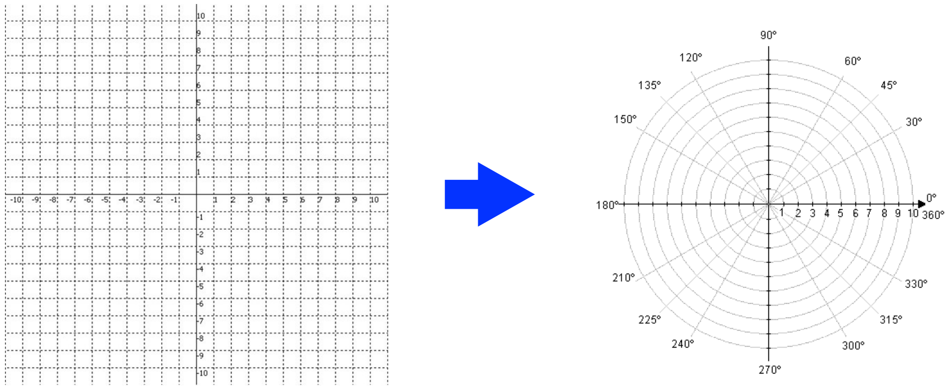
real imaginary

Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

how do we compute these?



polar transform

Recalling some basics

Complex numbers have two parts:

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real imaginary

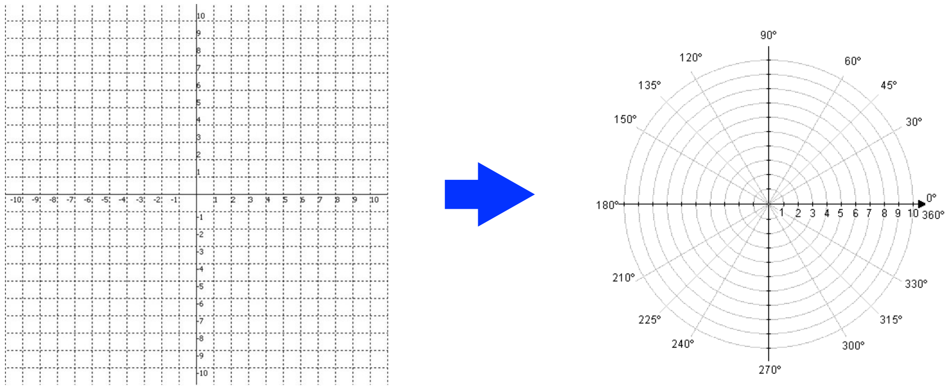
Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



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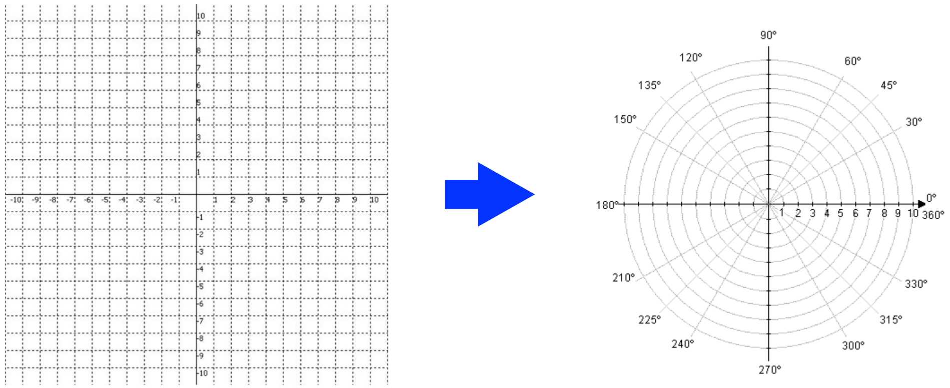


polar transform

How do you write these in exponential form?

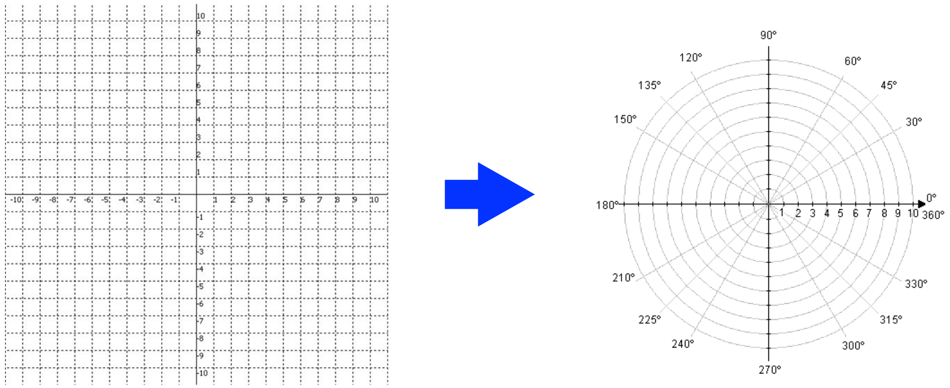
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

how did we get this?

exponential form

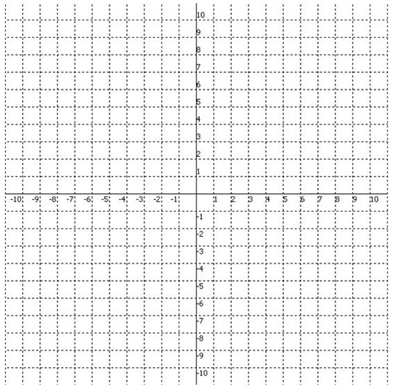
Recalling some basics

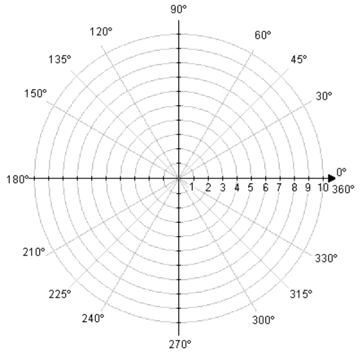
Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary





Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

exponential form

This will help us understand the Fourier transform equations

Fourier transform

Fourier transform | inverse Fourier transform

continuous

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi kx} \, dx$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi kx} \, dk$$

discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$

$k = 0, 1, 2, \ldots, N-1$

$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

$x = 0, 1, 2, \ldots, N-1$

Where is the connection to the *"summation of sine waves"* idea?

Fourier transform

Where is the connection to the *"summation of sine waves"* idea?

$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j 2\pi k x / N}$$

Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

sum over frequencies

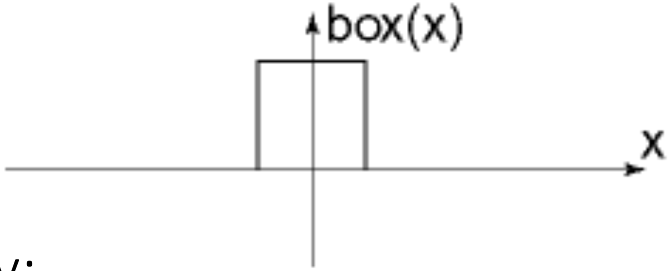
$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) \left\{ \cos\left(\frac{2\pi k x}{N}\right) + j \sin\left(\frac{2\pi k x}{N}\right) \right\}$$

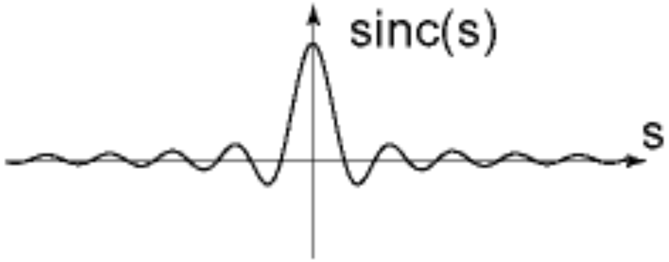
scaling parameter

wave components

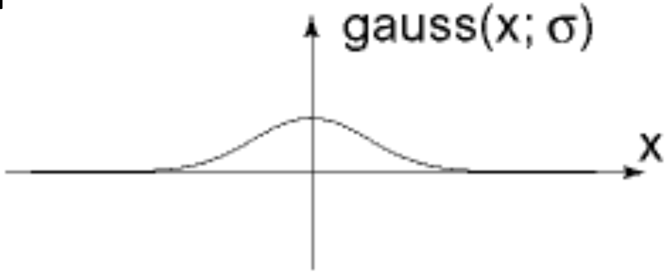
Fourier transform pairs

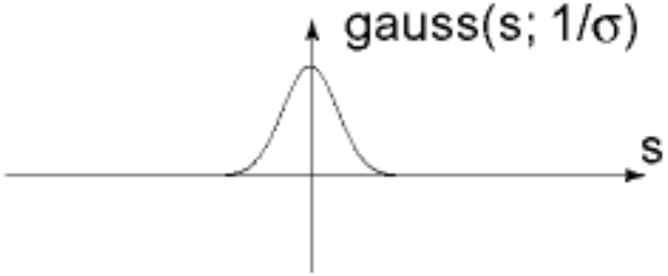
spatial domain | frequency domain

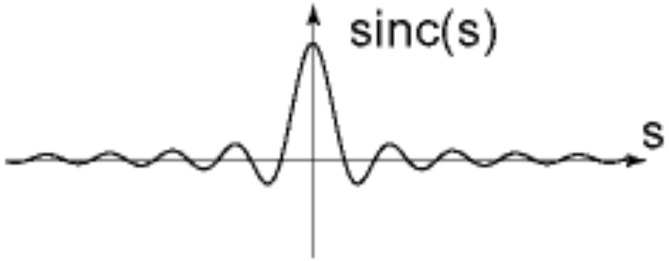


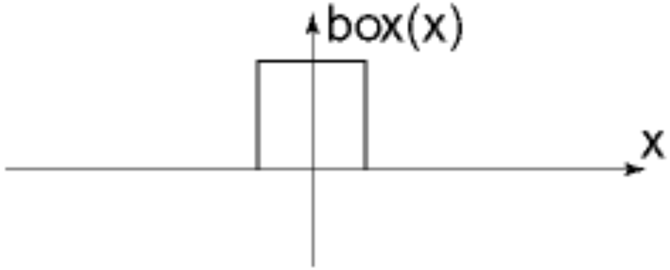


Note the symmetry: duality property of Fourier transform









Computing the discrete Fourier transform (DFT)

$$F(k) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$ is just a matrix multiplication:

$$\boldsymbol{F} = \boldsymbol{W}\boldsymbol{f}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix} \qquad W = e^{-j2\pi/N}$$

In practice this is implemented using the *fast Fourier transform* (FFT) algorithm.

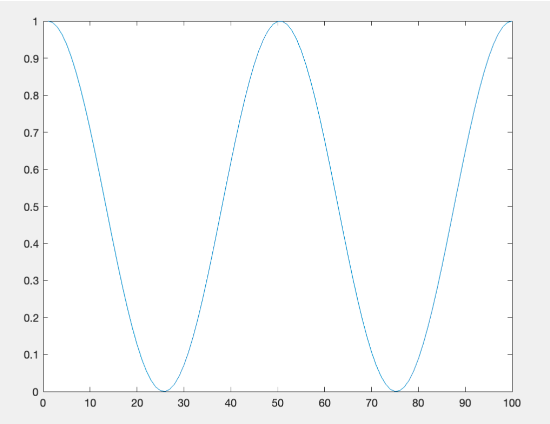
2D Frequency Analysis

Examples

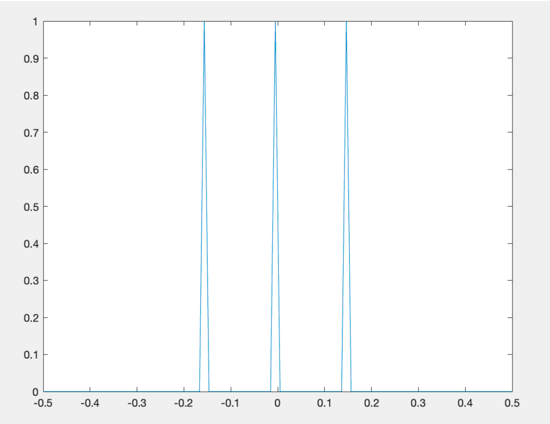
Spatial domain visualization

Frequency domain visualization

1D



$|F(k)|$

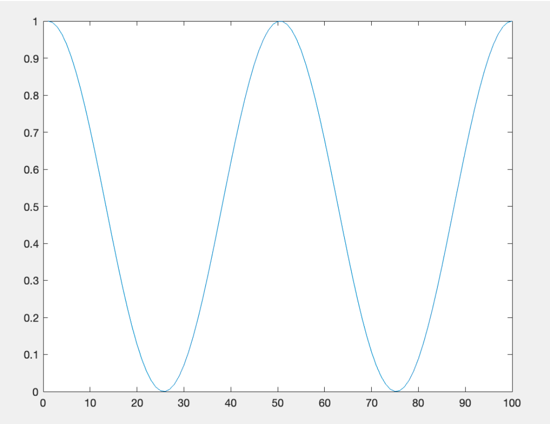


Examples

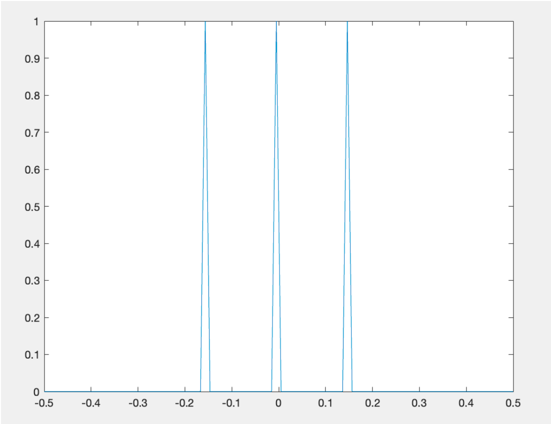
Spatial domain visualization

Frequency domain visualization

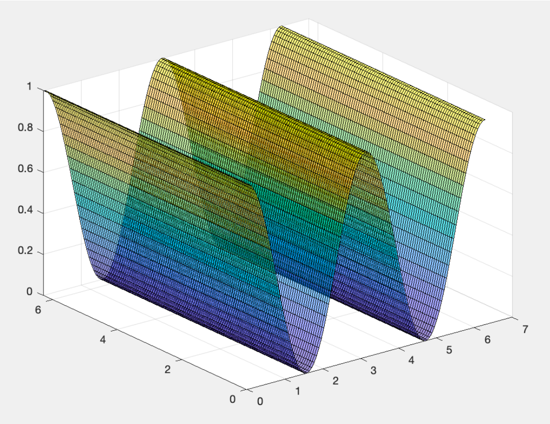
1D



$|F(k)|$



2D



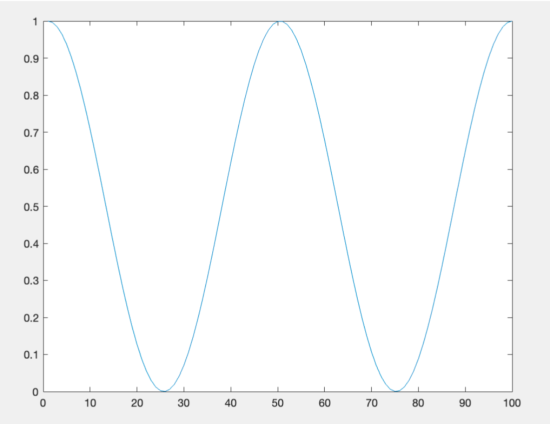
?

Examples

Spatial domain visualization

Frequency domain visualization

1D



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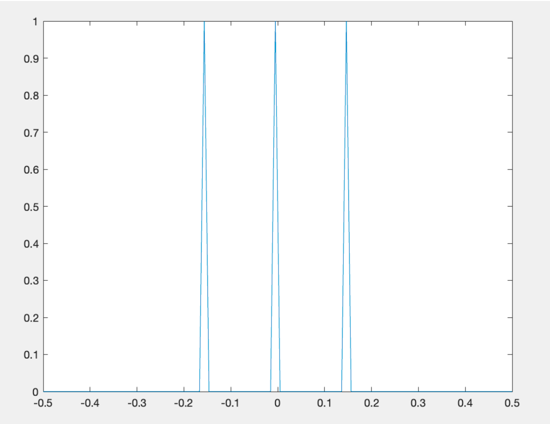
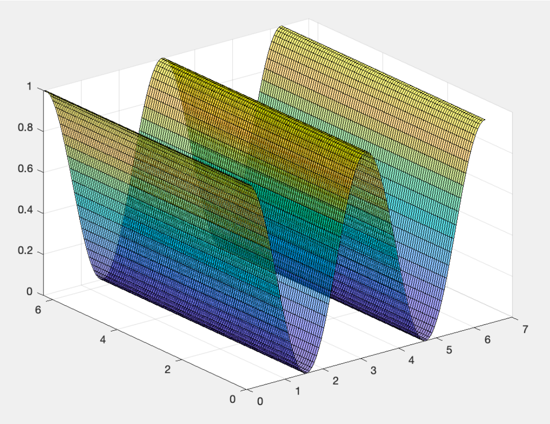
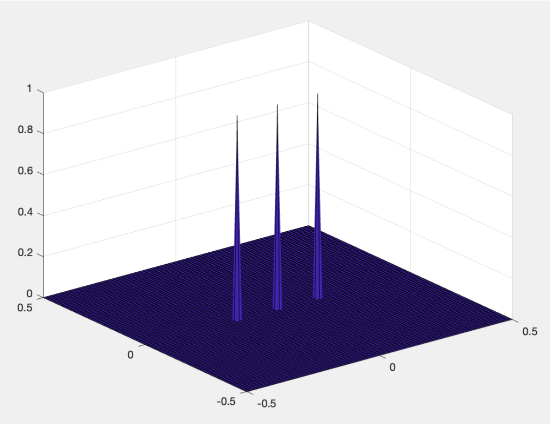


2D



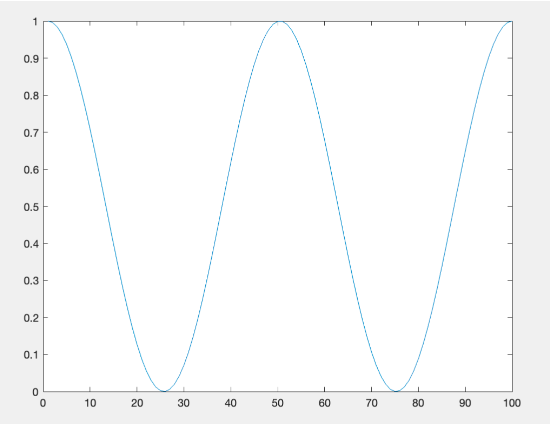


Examples

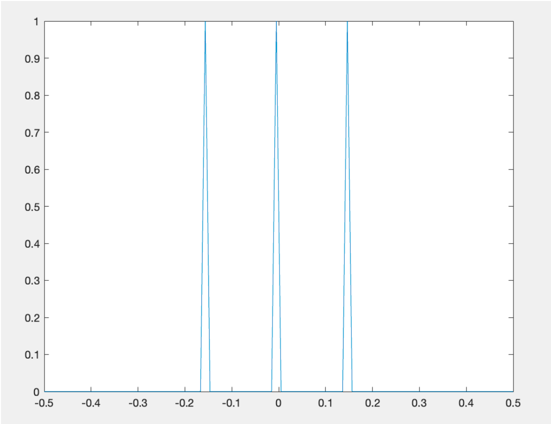
Spatial domain visualization

Frequency domain visualization

1D

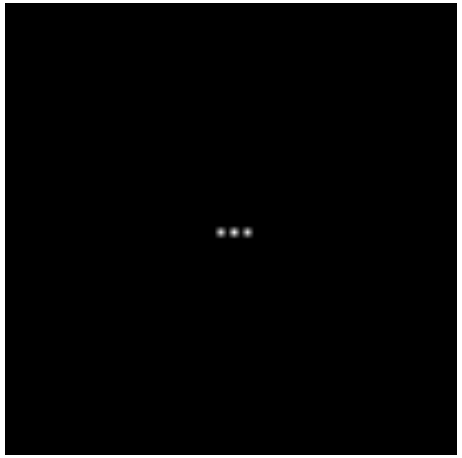


$|F(k)|$



2D

k_y

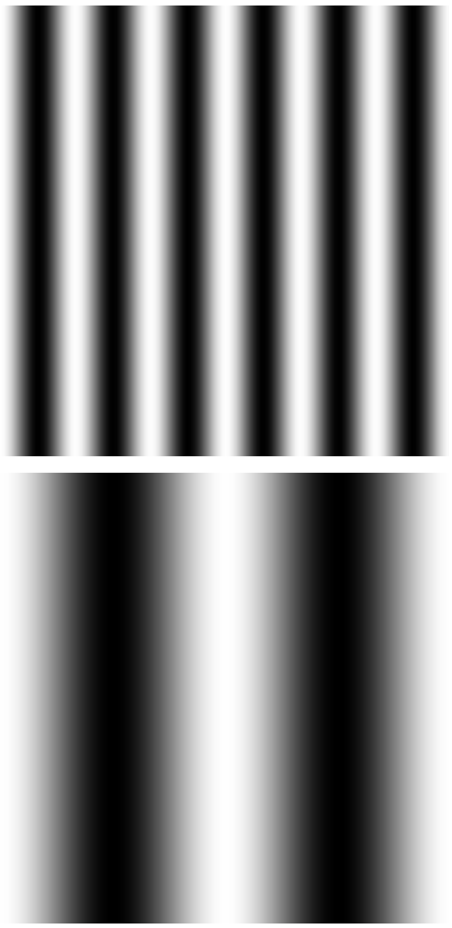


k_x

What do the three dots correspond to?

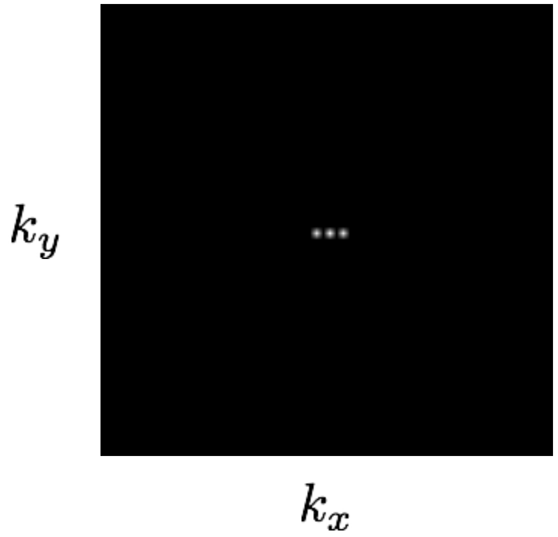
Examples

Spatial domain visualization



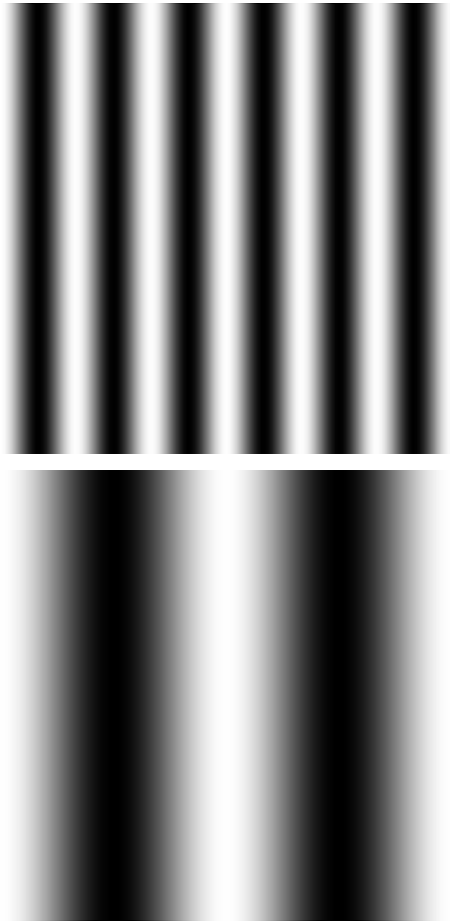
Frequency domain visualization

?

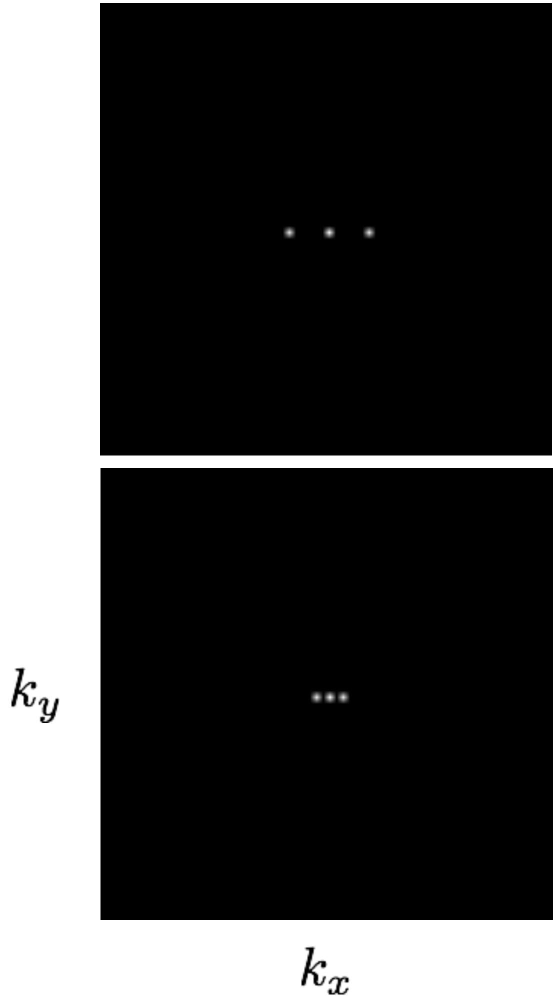


Examples

Spatial domain visualization

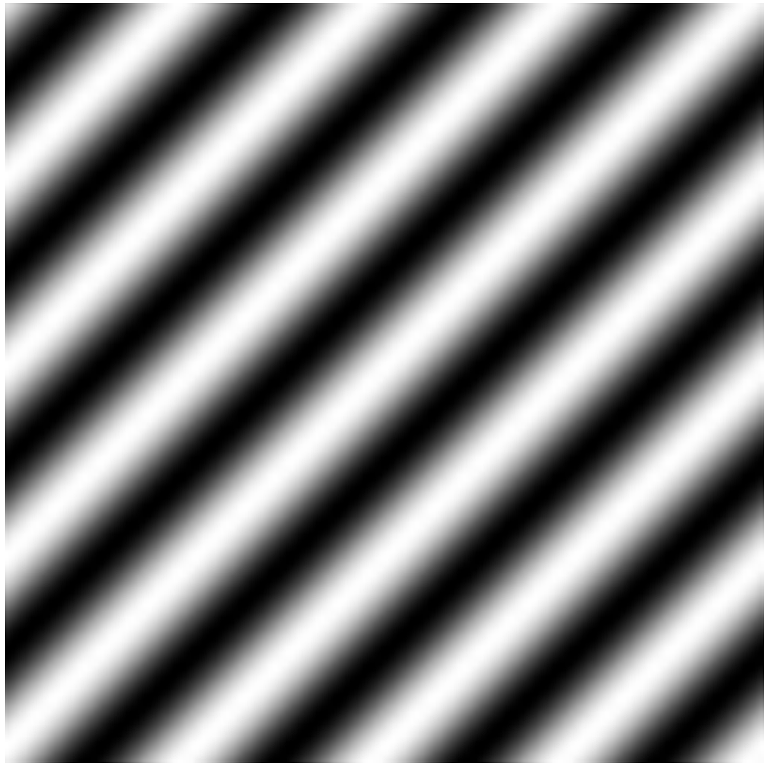


Frequency domain visualization



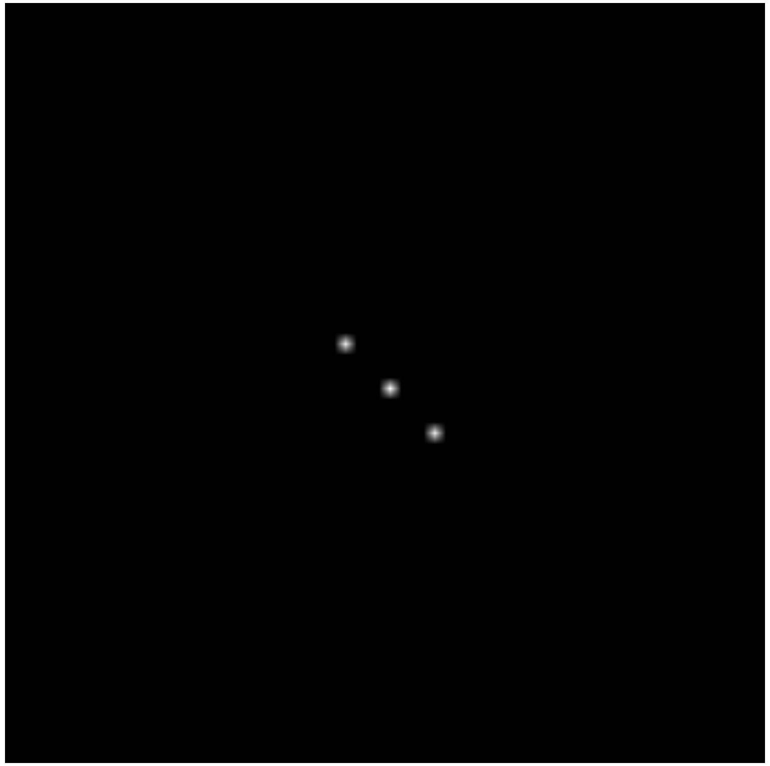
Examples

How would you generate this image with sine waves?



Examples

How would you generate this image with sine waves?



Has both an x and y components

Examples

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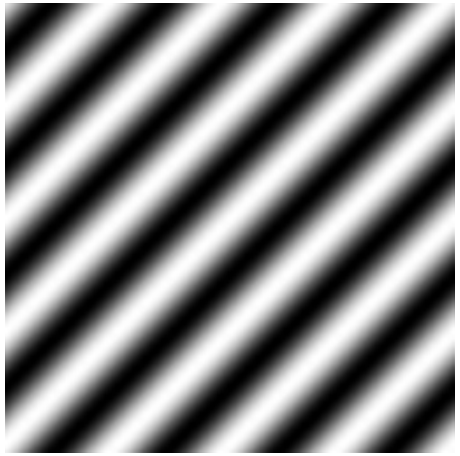

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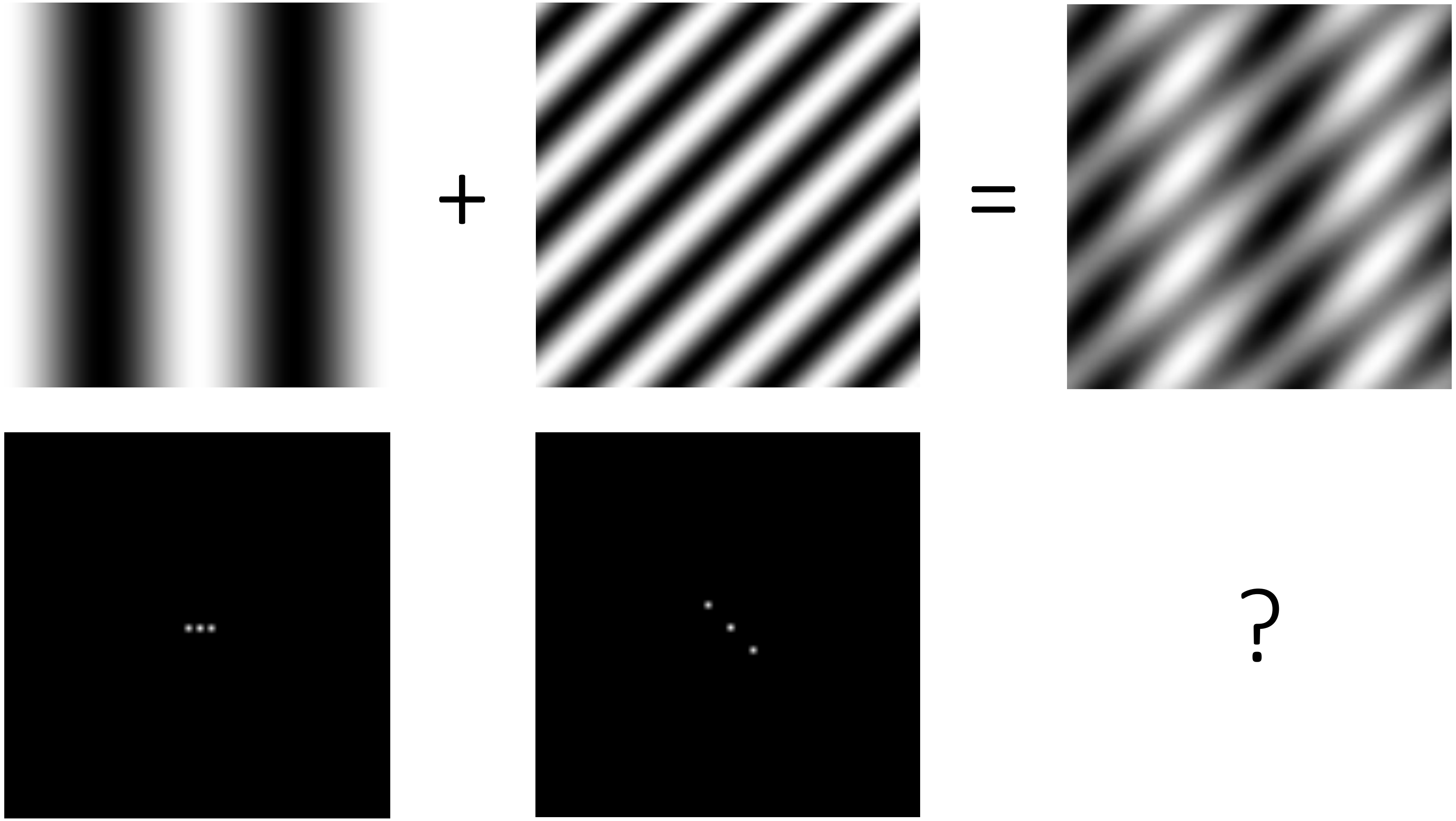
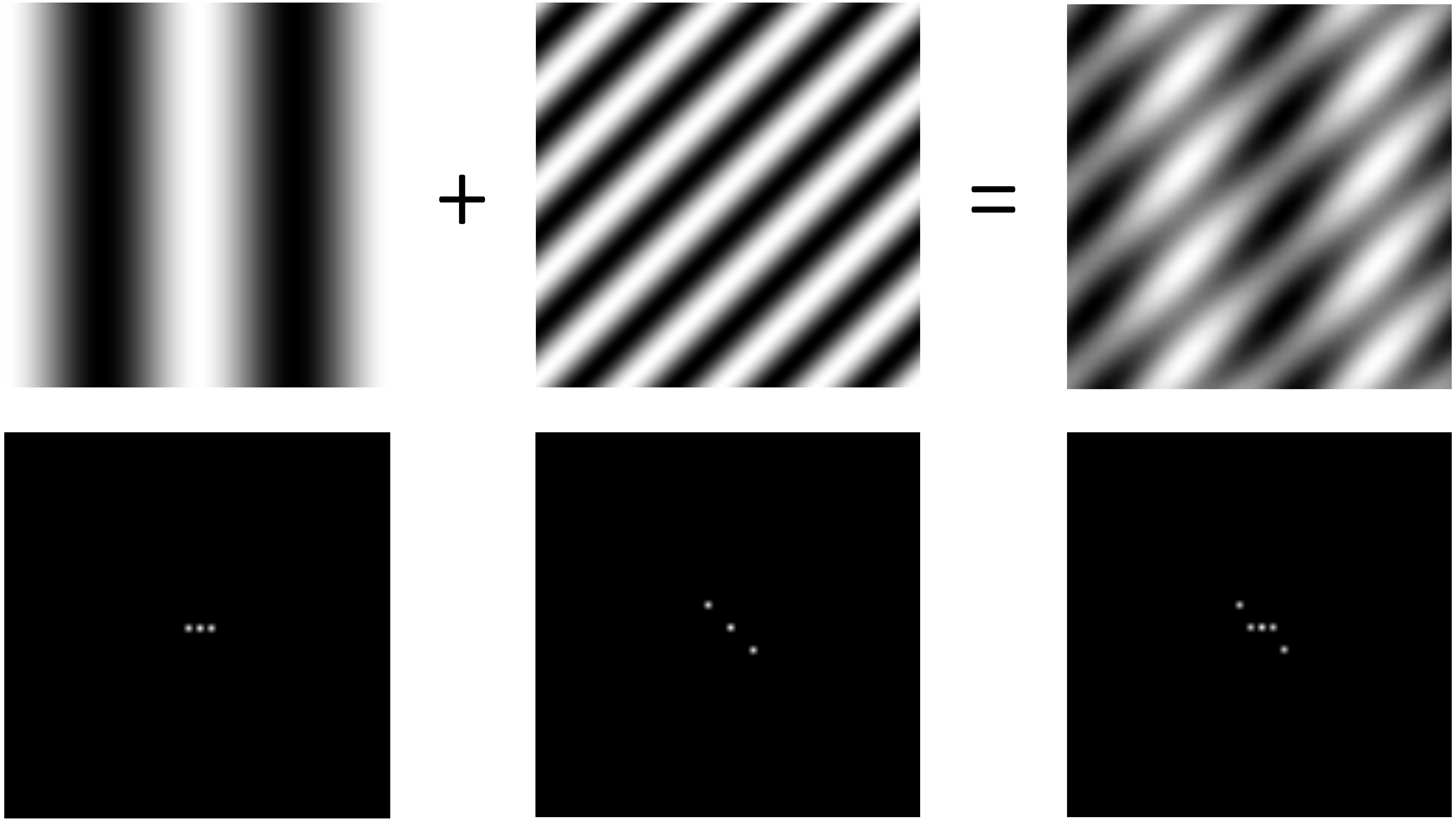
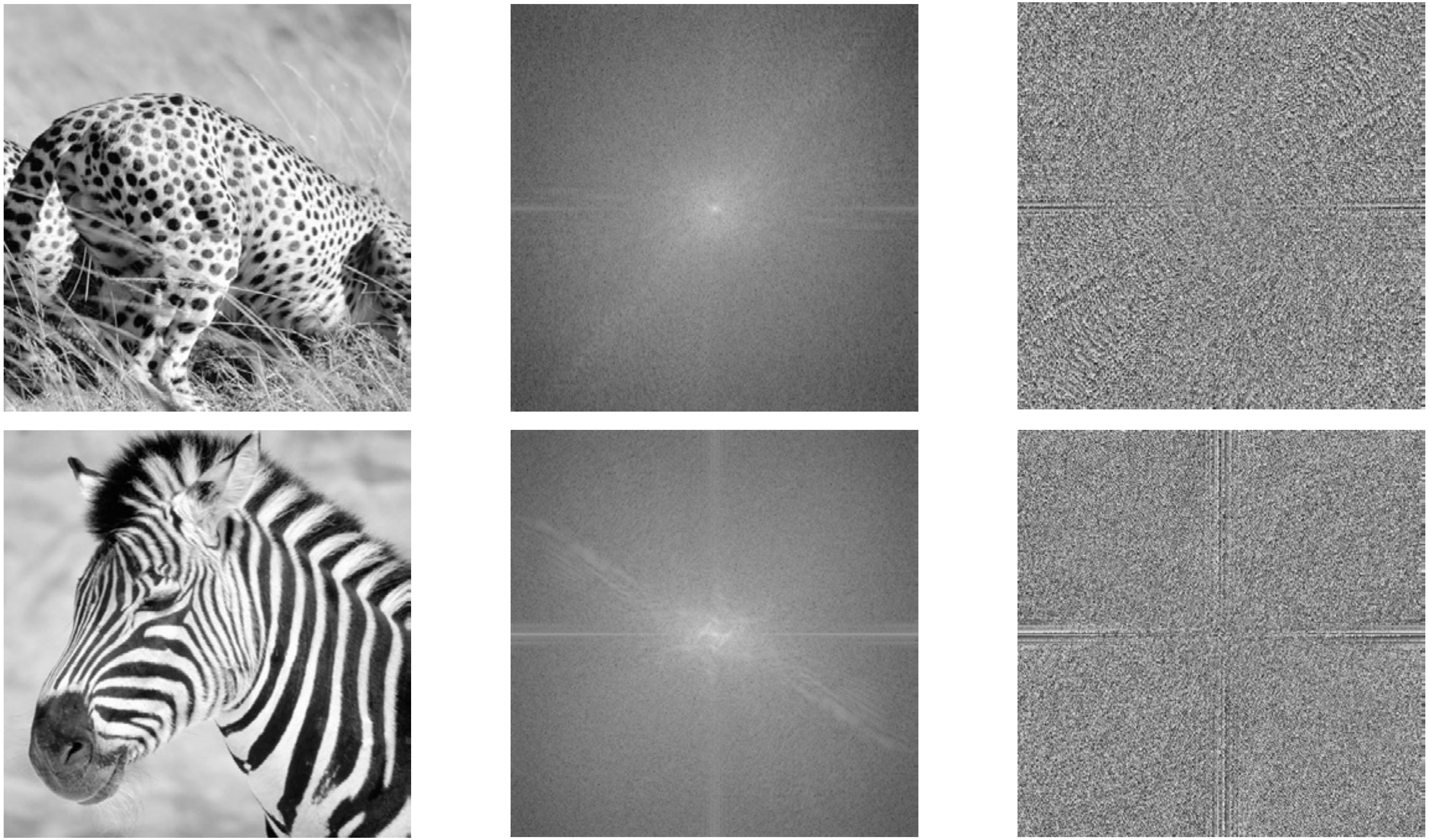
Examples



Examples



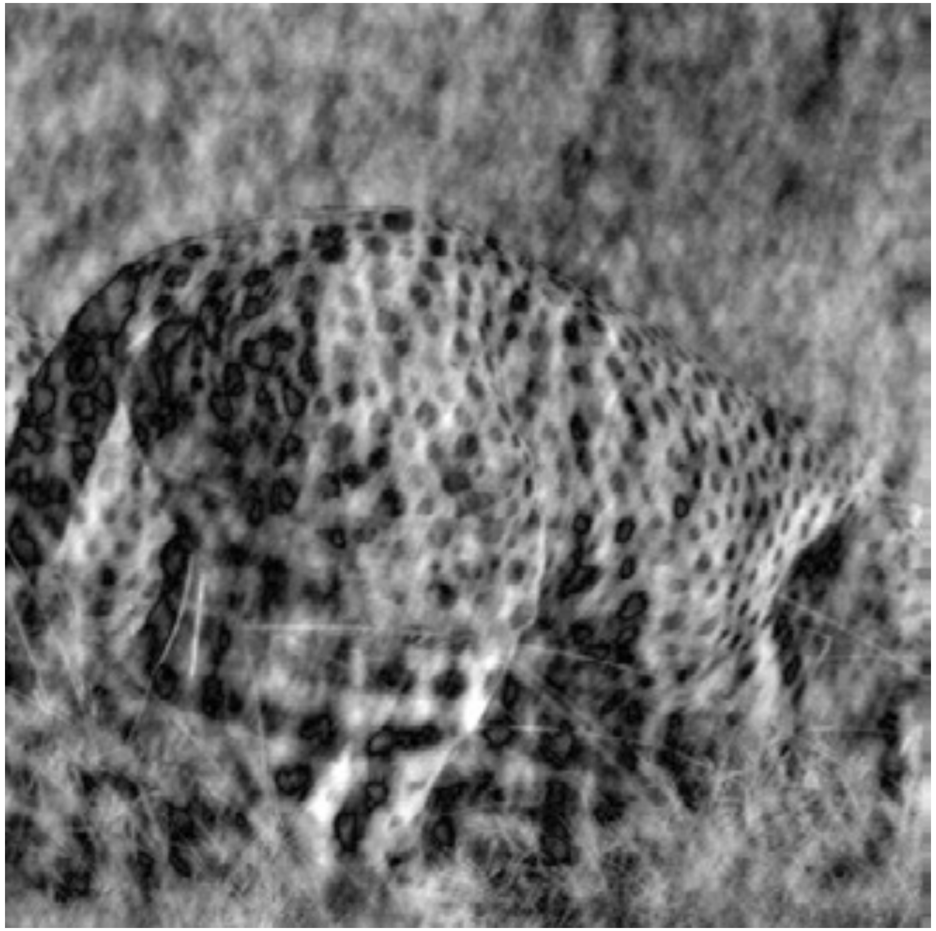
Fourier transforms of natural images



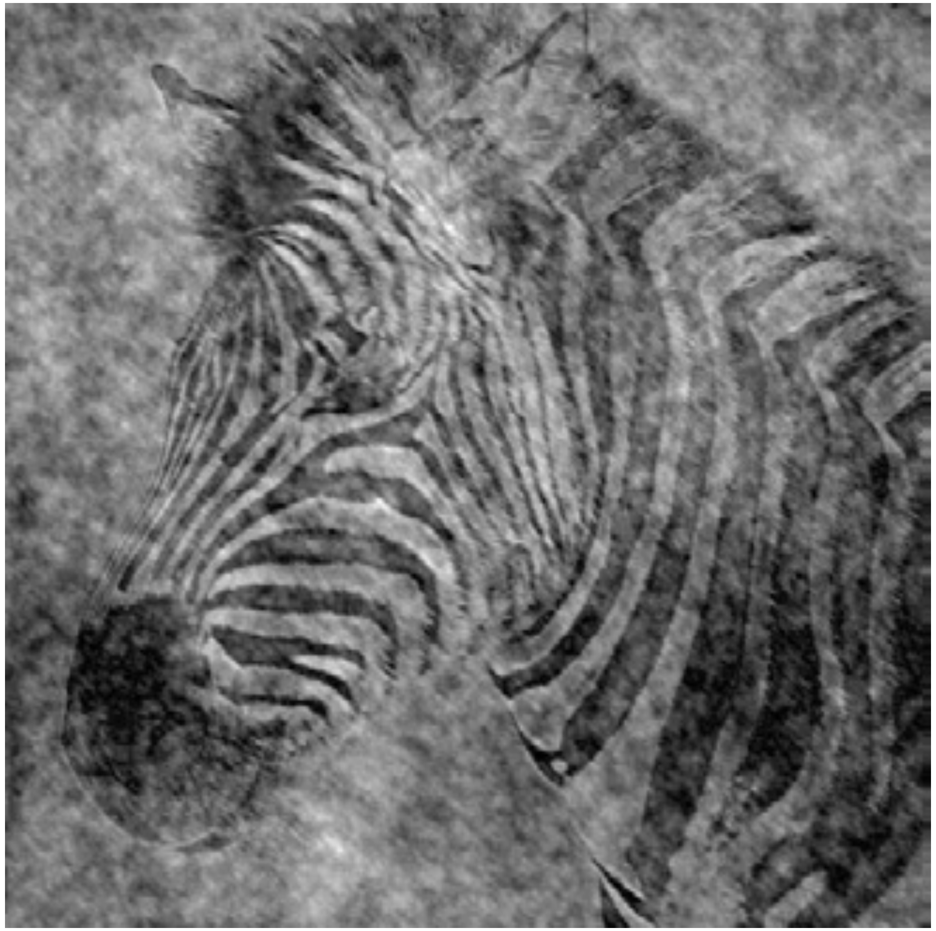
original amplitude phase

Fourier transforms of natural images

Image phase matters!



cheetah phase with zebra amplitude



zebra phase with cheetah amplitude

Frequency-domain filtering

The convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!

What do we use convolution for?

Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution:

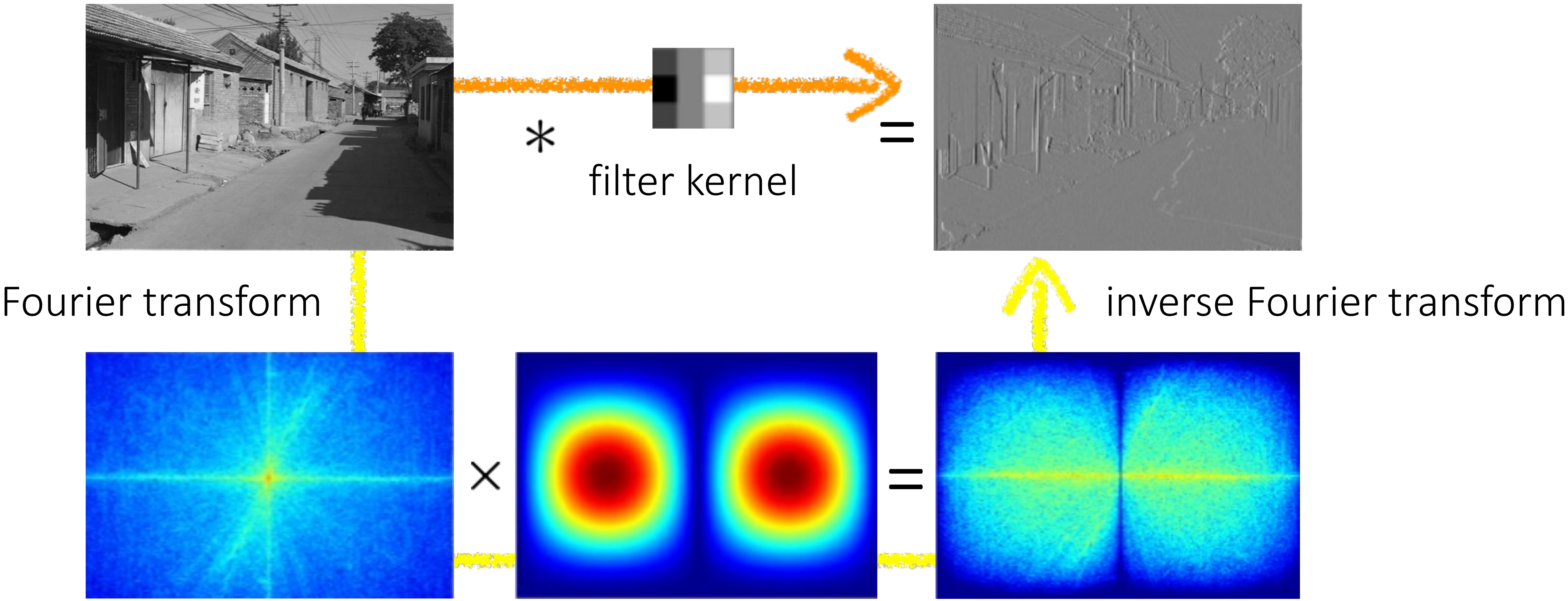
$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

filtered signal ↗ ... ↖ filter ↖ input signal

Using the convolution theorem, we can interpret and implement all types of linear shift-invariant filtering as multiplication in frequency domain.

Why implement convolution in frequency domain?

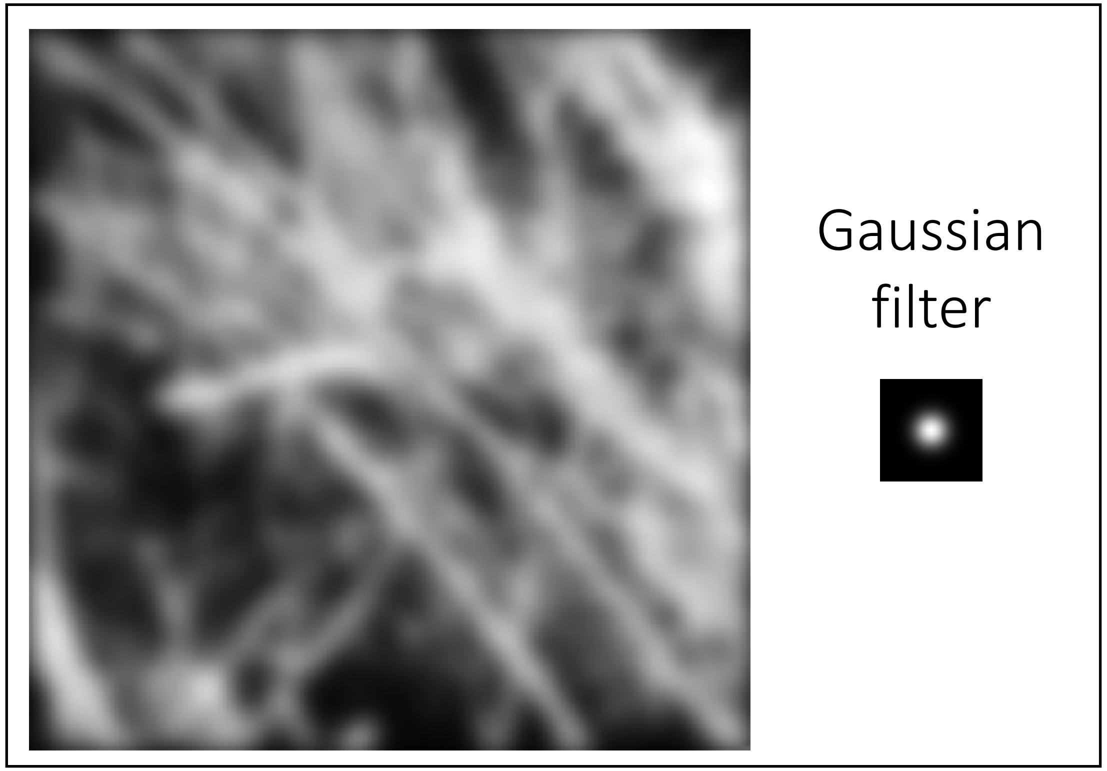
Spatial domain filtering



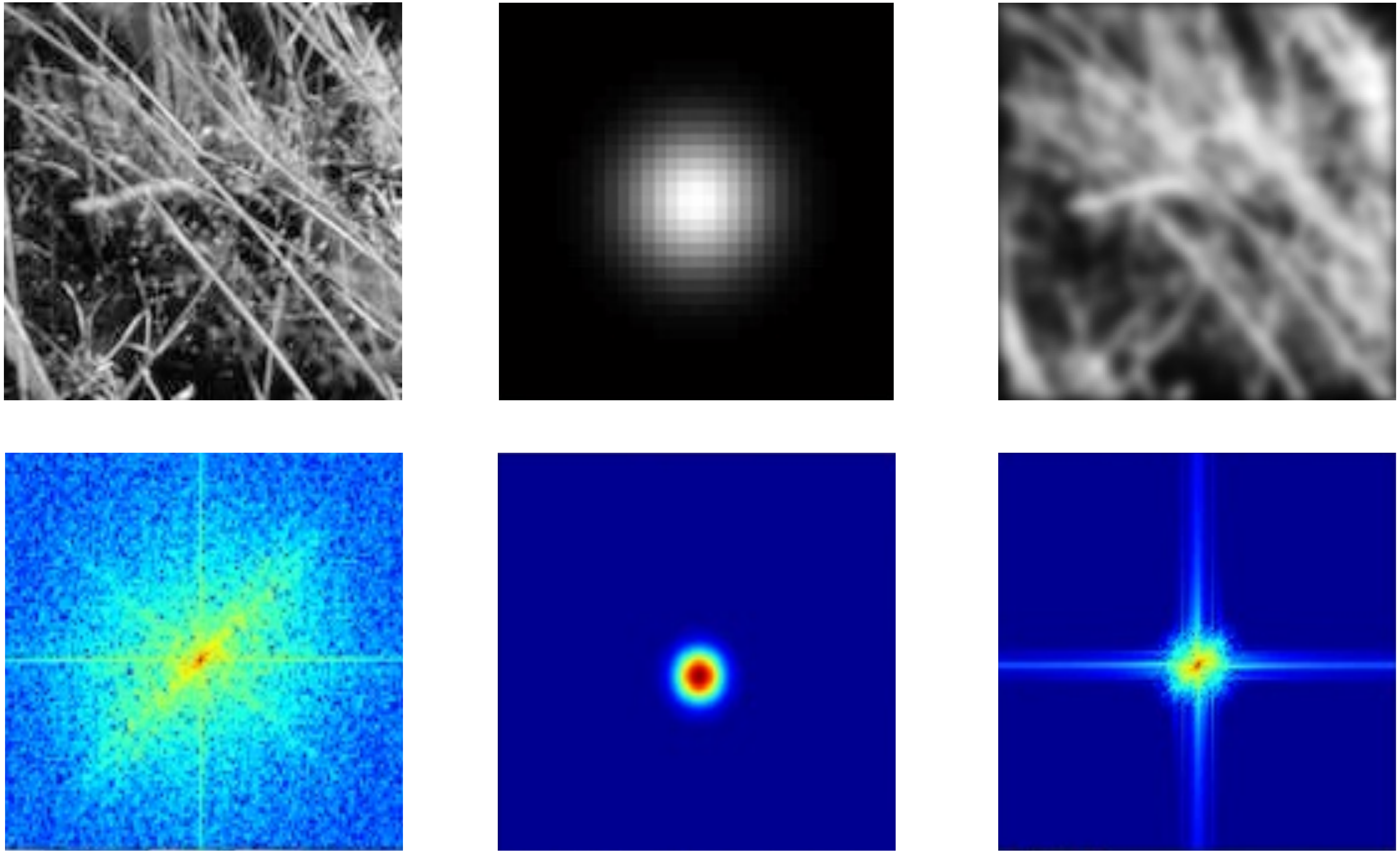
Frequency domain filtering

Revisiting blurring

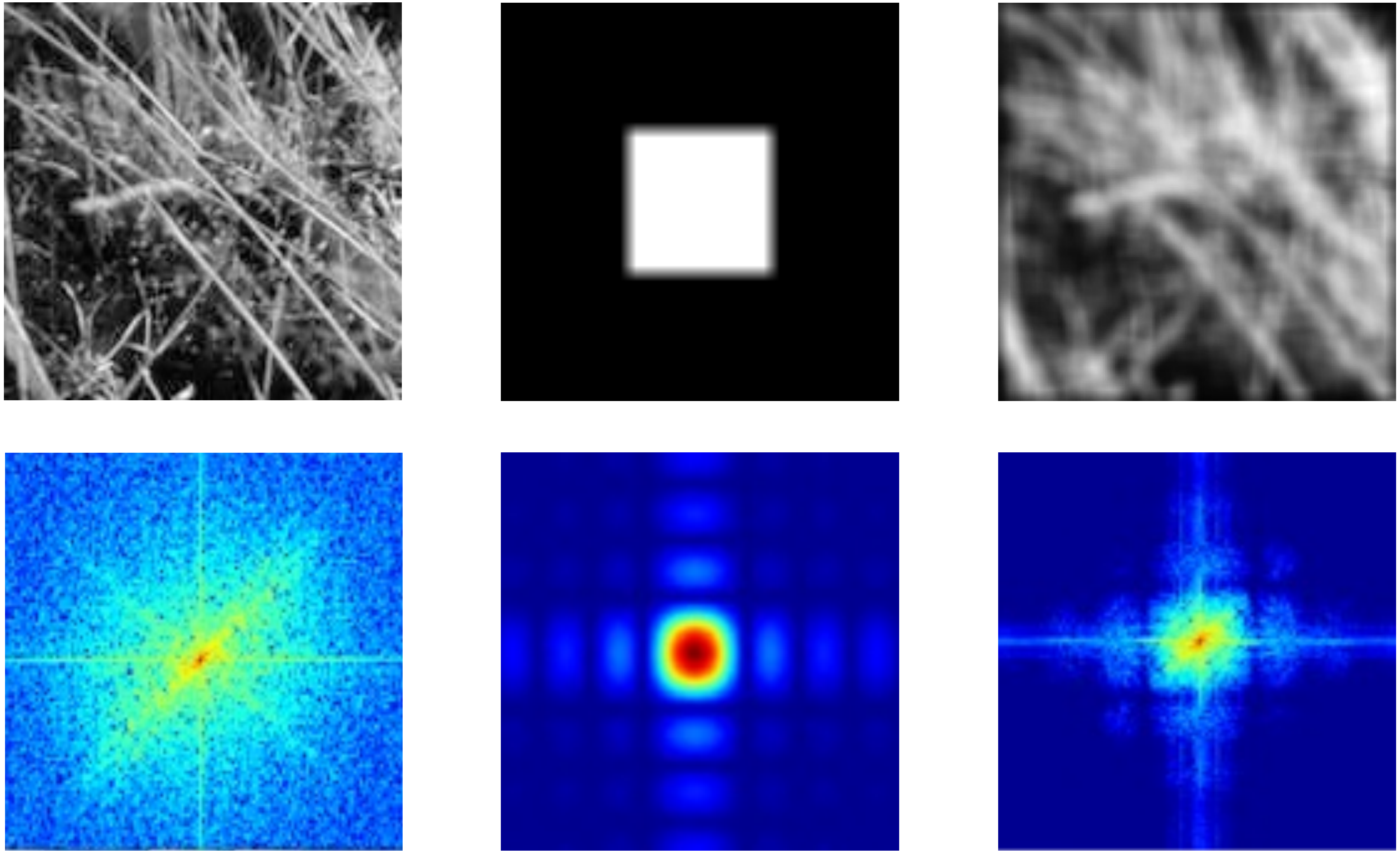
Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?



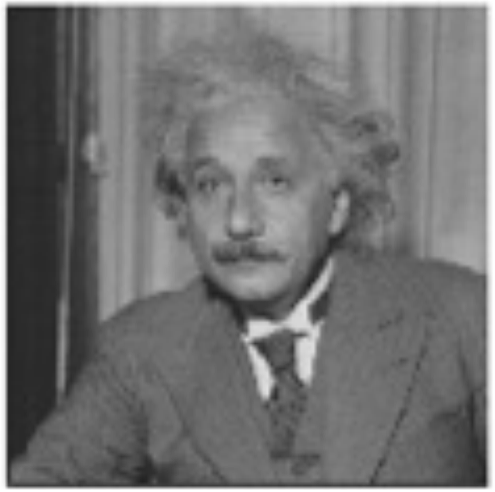
Gaussian blur



Box blur



More filtering examples



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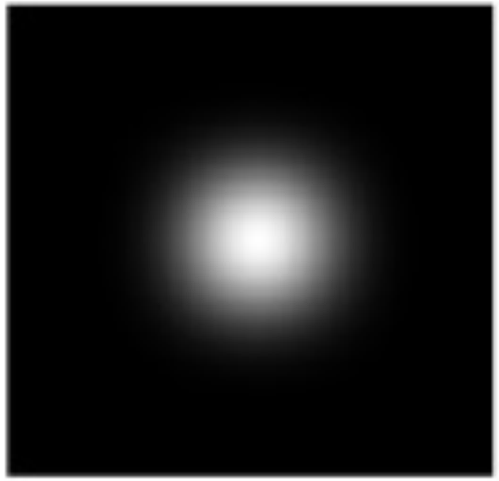

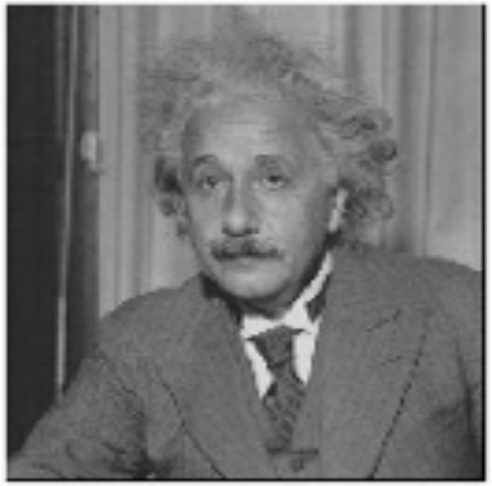

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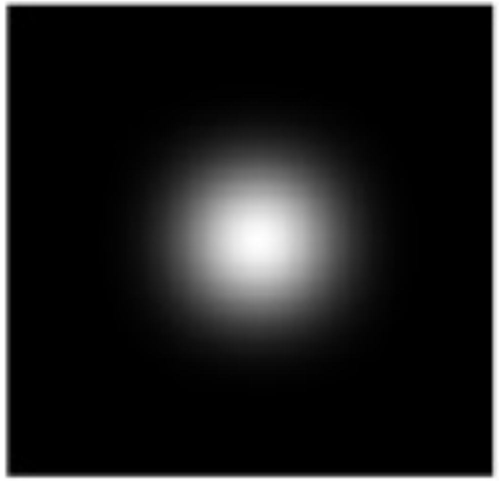
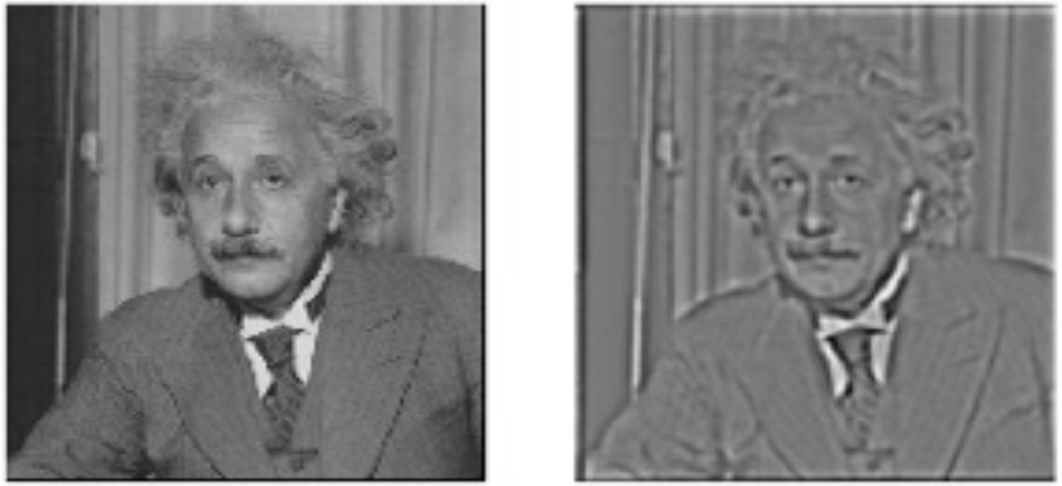
filters shown in frequency-domain

More filtering examples

low-pass





band-pass

filters shown in frequency-domain

More filtering examples

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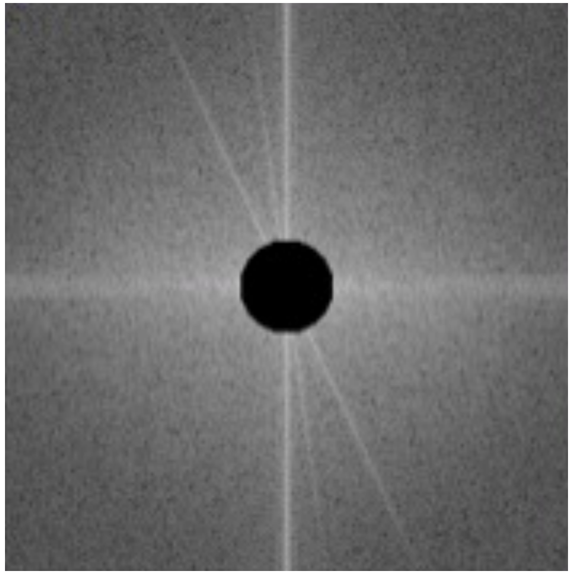
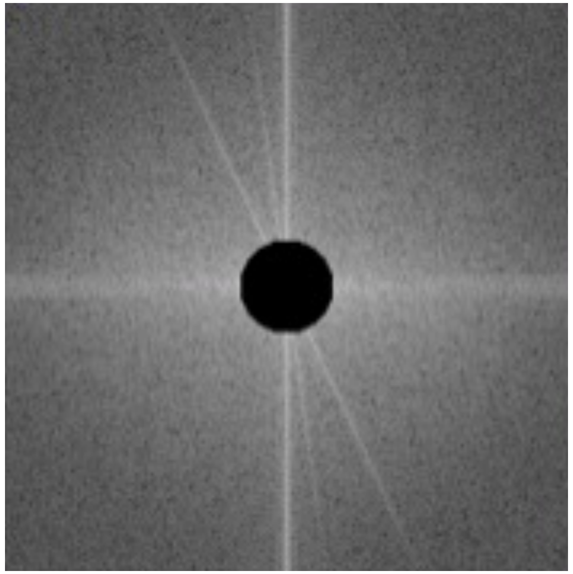
high-pass



More filtering examples

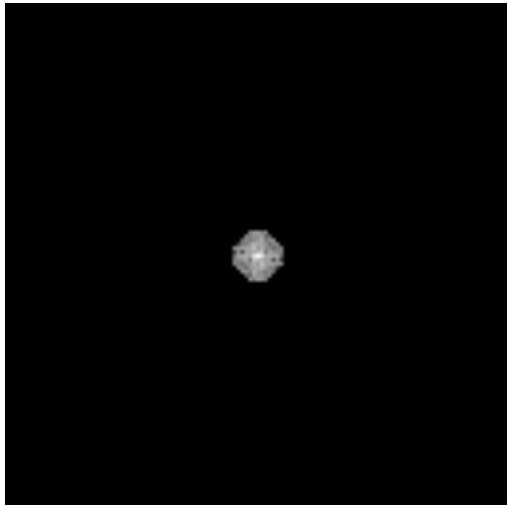
high-pass



More filtering examples

original image

low-pass filter



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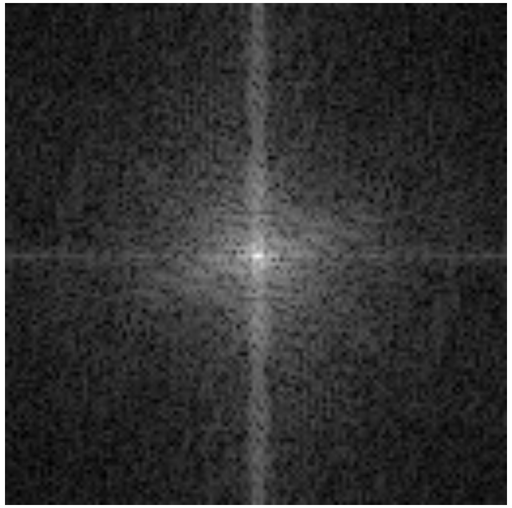
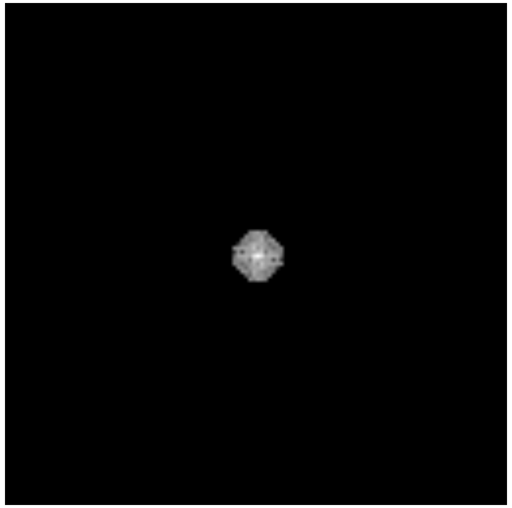
frequency magnitude



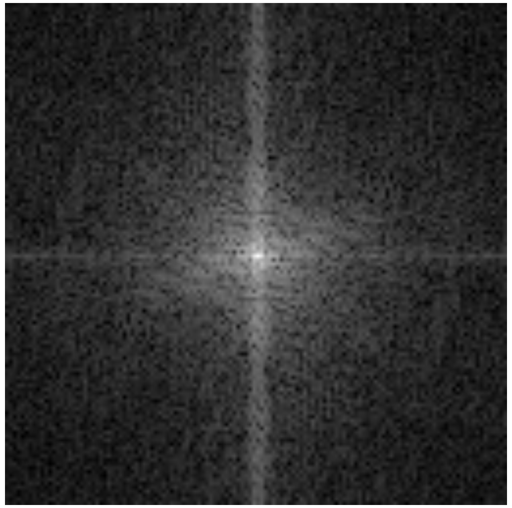
More filtering examples

original image

low-pass filter



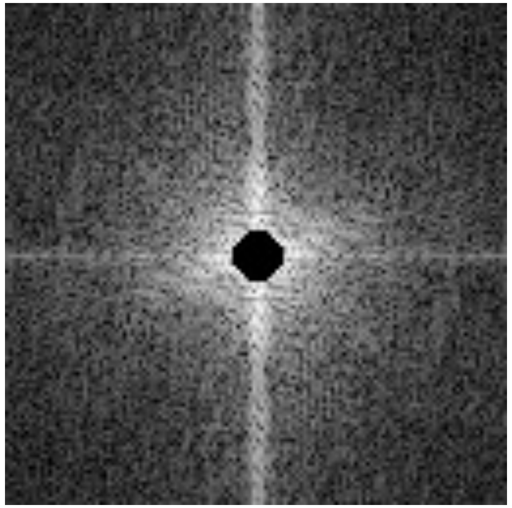
frequency magnitude



More filtering examples

original image

high-pass filter



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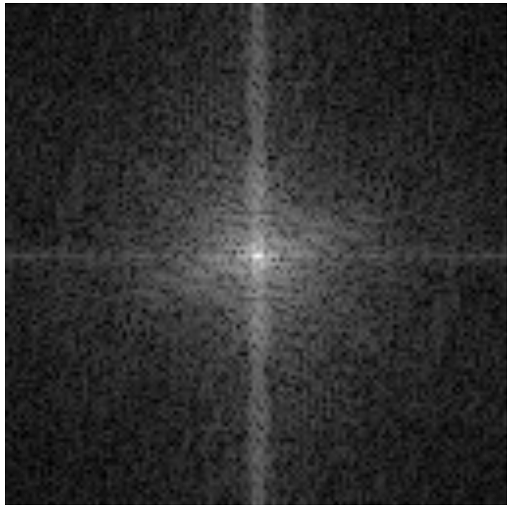
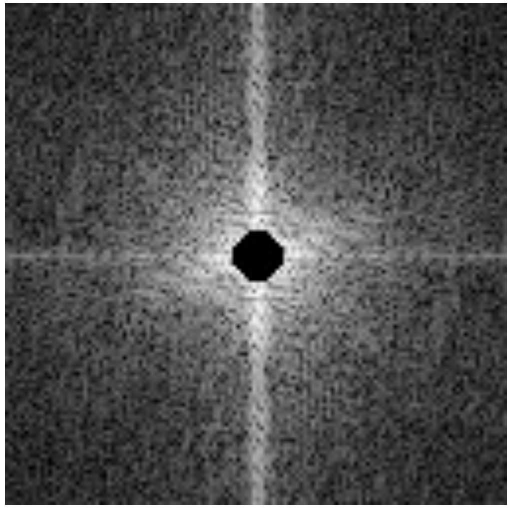
frequency magnitude



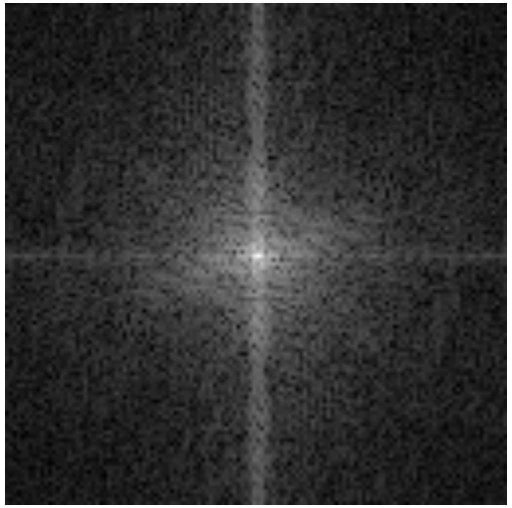
More filtering examples

original image

high-pass filter



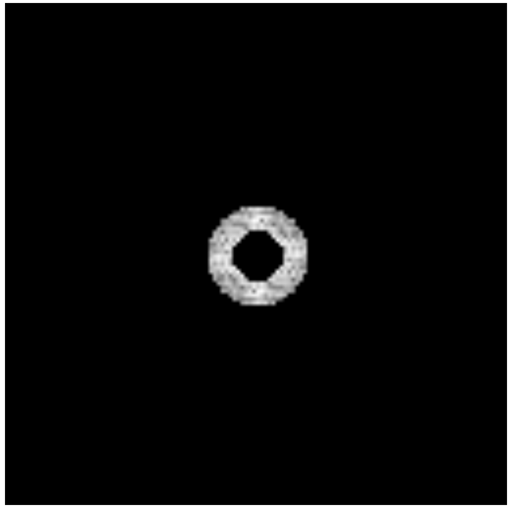
frequency magnitude

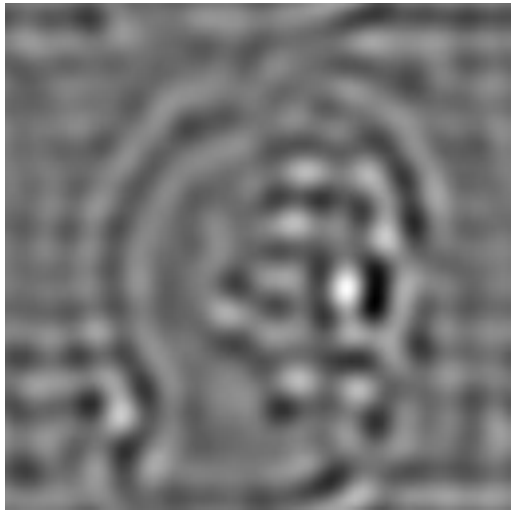


More filtering examples

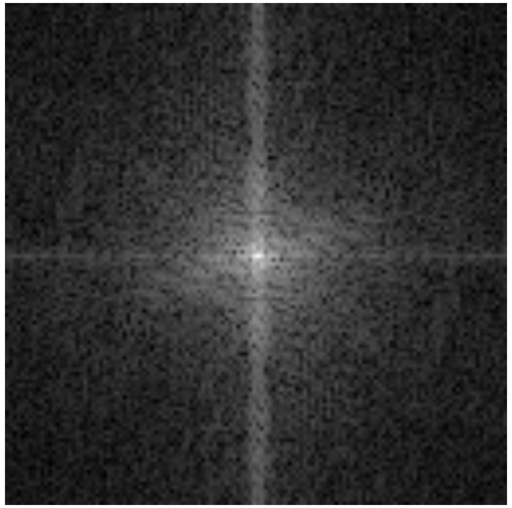
original image

band-pass filter





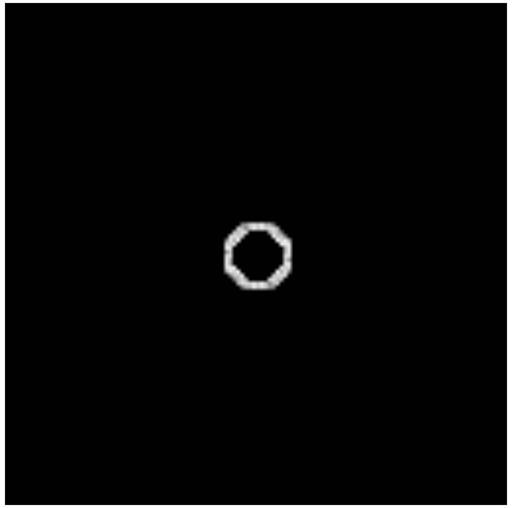
frequency magnitude



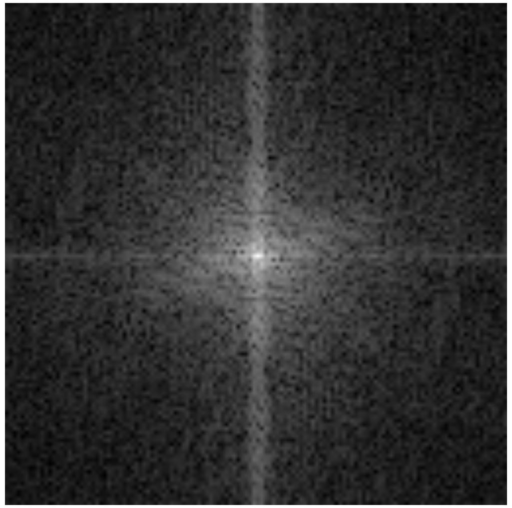
More filtering examples

original image

band-pass filter



frequency magnitude

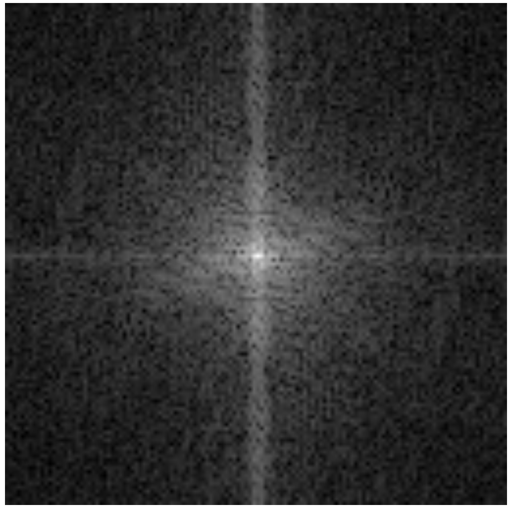


More filtering examples

original image

band-pass filter

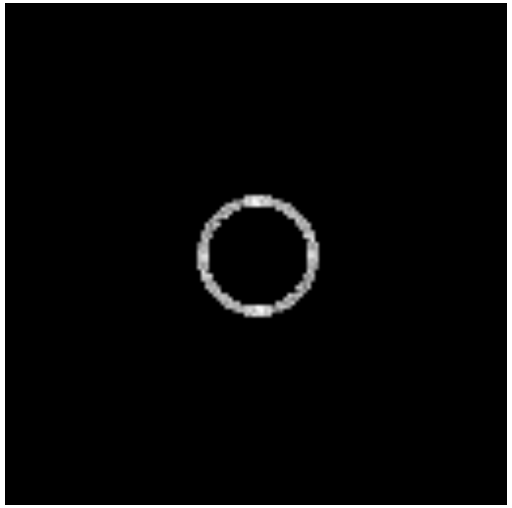
frequency magnitude



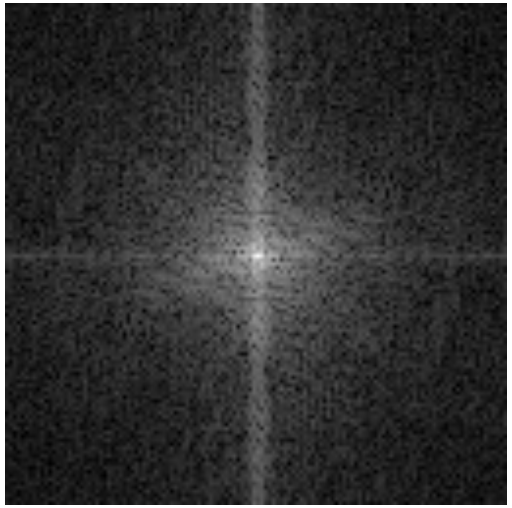
More filtering examples

original image

band-pass filter



frequency magnitude



Revisiting sampling

The Nyquist-Shannon sampling theorem

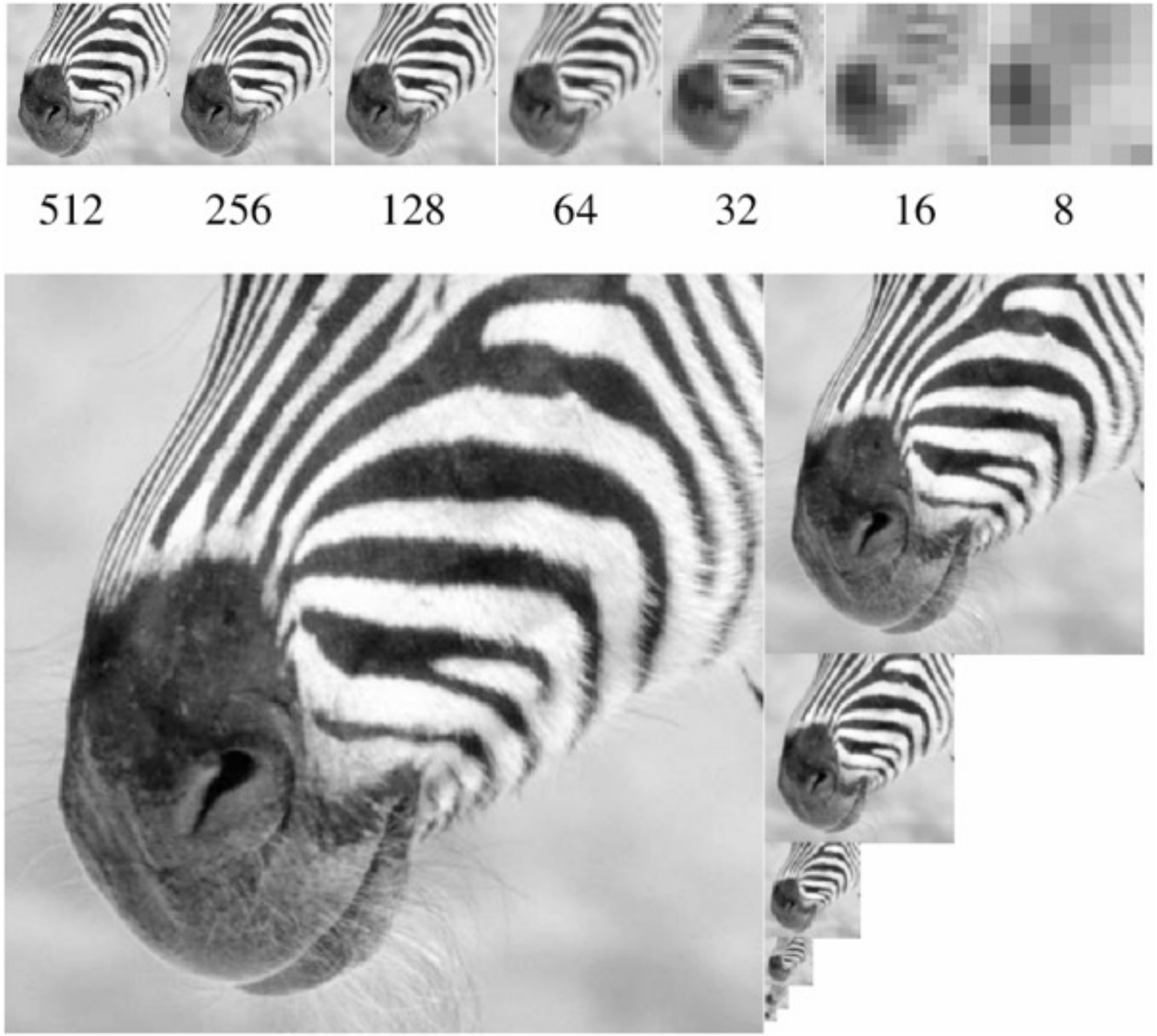
A continuous signal can be perfectly reconstructed from its discrete version using linear interpolation, if sampling occurred with frequency:

$$f_s \geq 2 f_{\max}$$

← This is called the Nyquist frequency

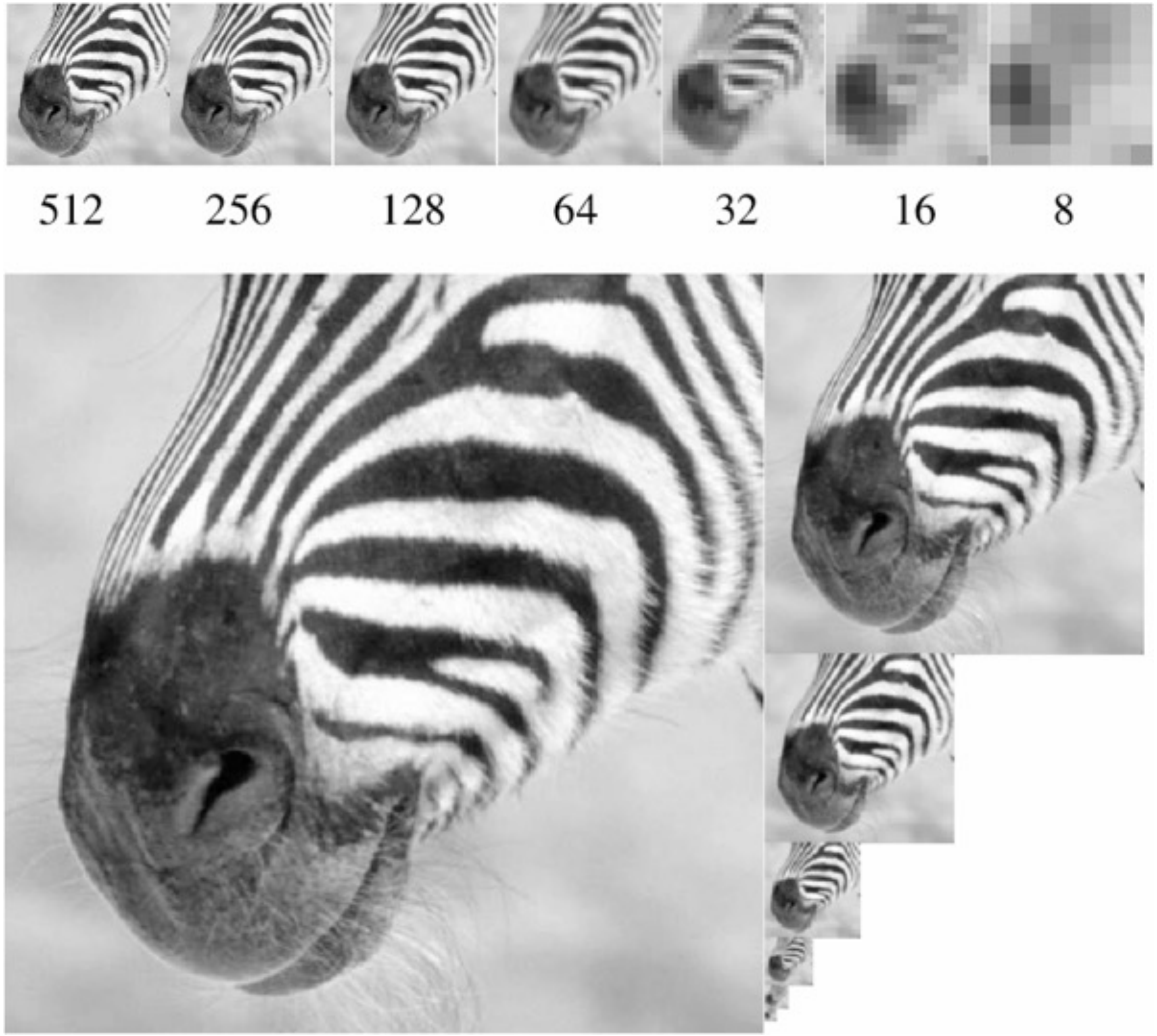
Equivalent reformulation: When downsampling, aliasing does not occur if samples are taken at the Nyquist frequency or higher.

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

- Gaussian blurring is low-pass filtering.
- By blurring we try to sufficiently decrease the Nyquist frequency to avoid aliasing.

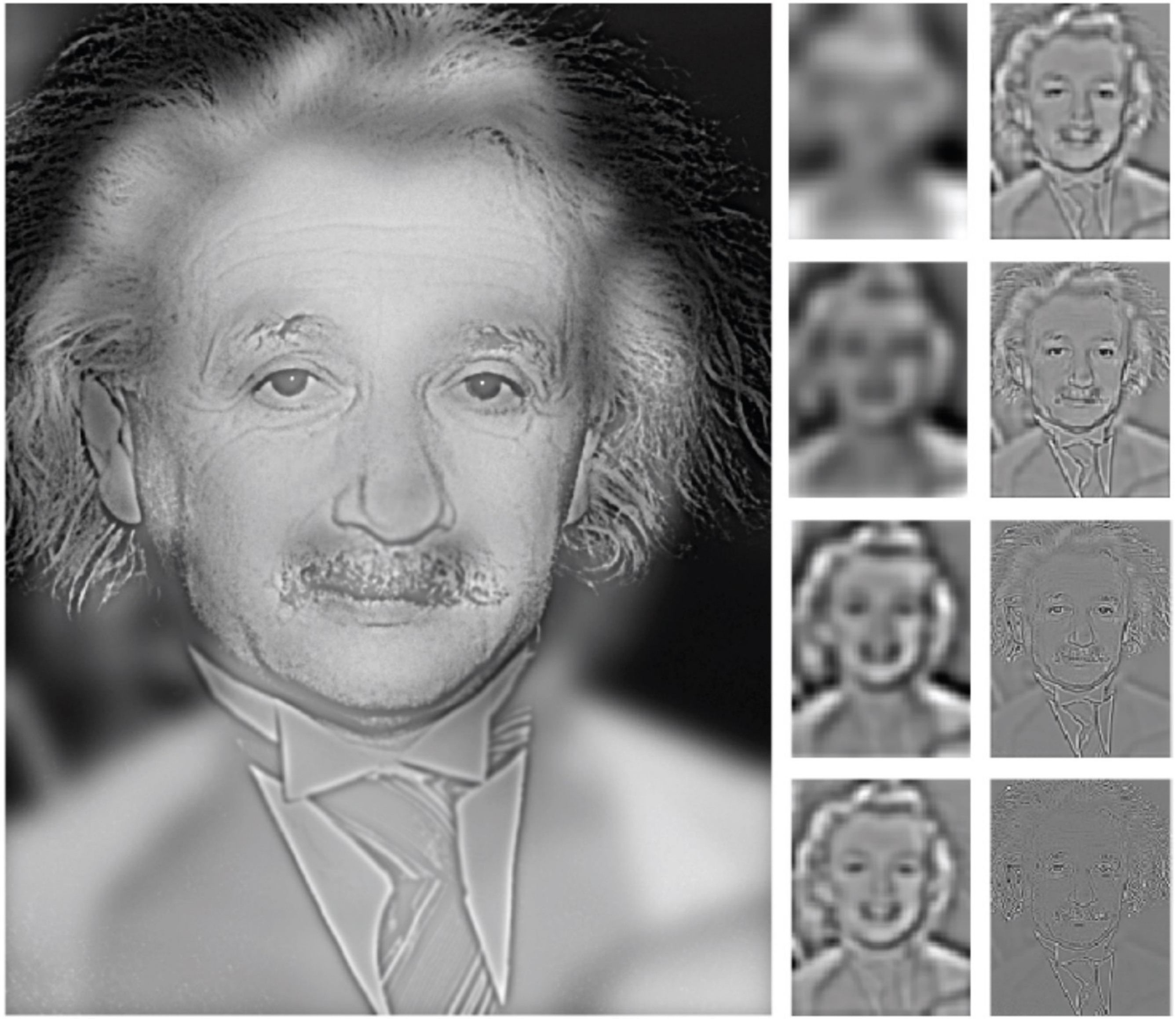
How large should the Gauss blur we use be?

Frequency-domain filtering in human vision

"Hybrid image"

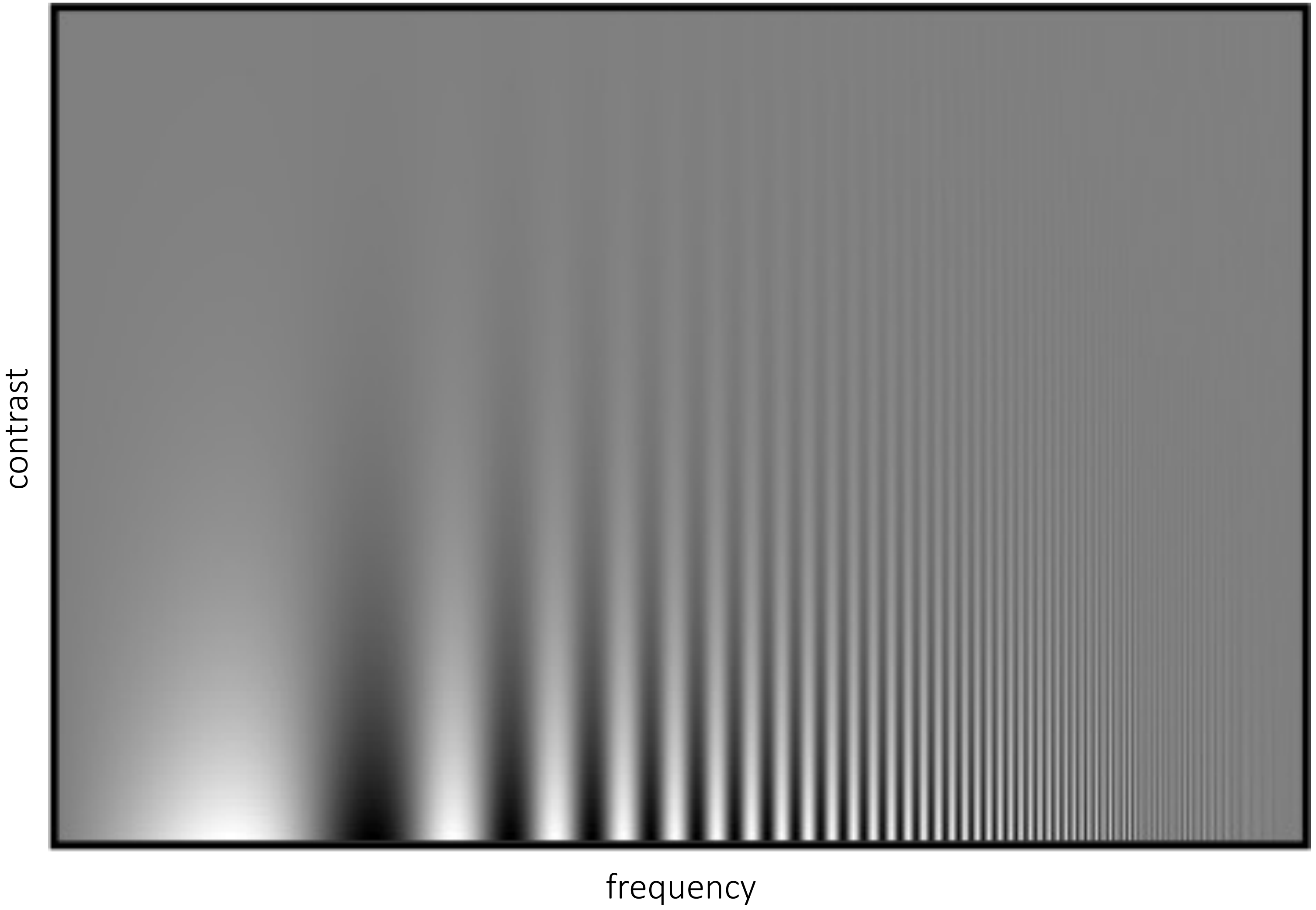
Aude Oliva and Philippe Schyns

Frequency-domain filtering in human vision



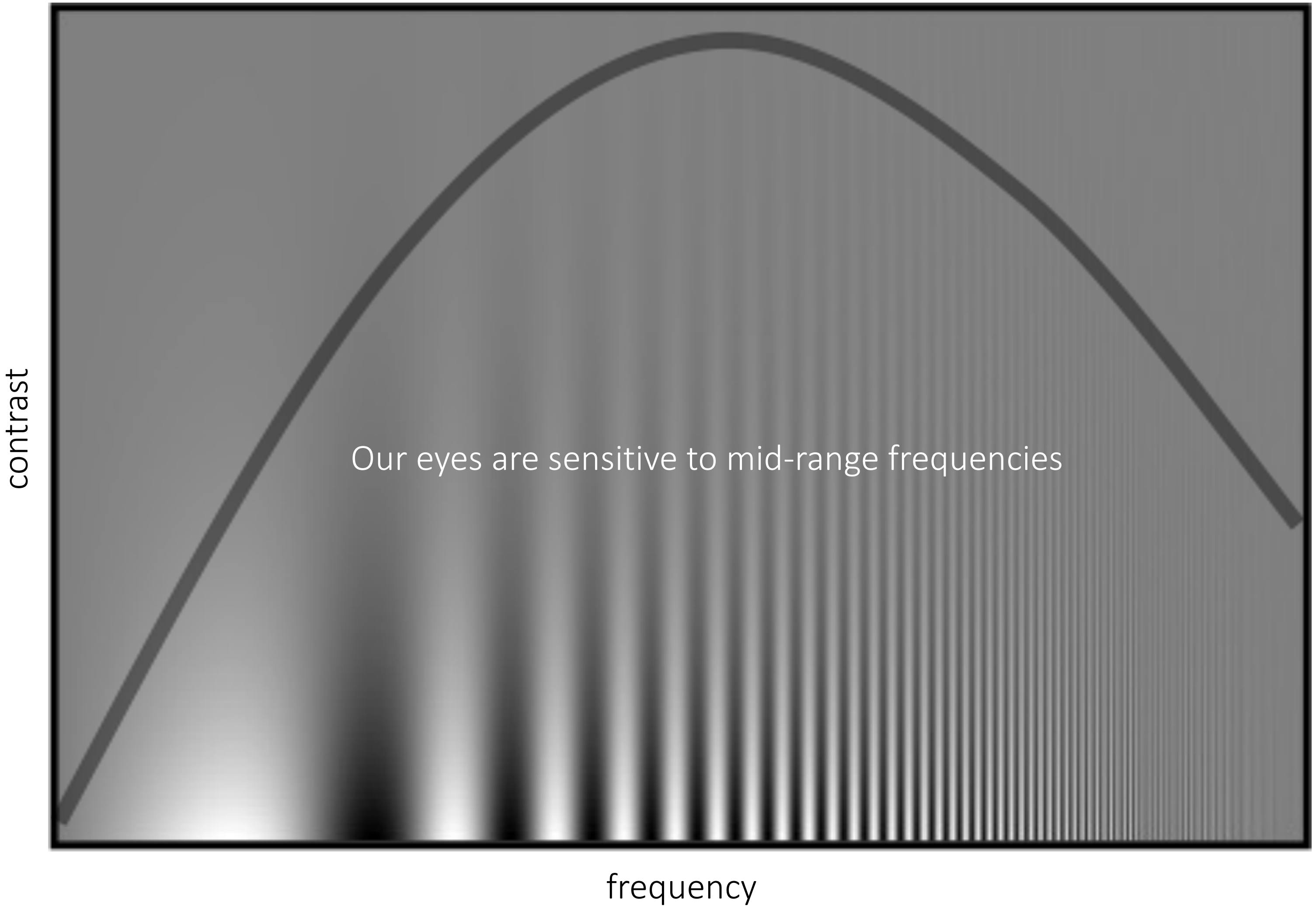
Variable frequency sensitivity

Experiment: Where do you see the stripes?



Variable frequency sensitivity

Campbell-Robson contrast sensitivity curve



- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid frequencies dominate perception