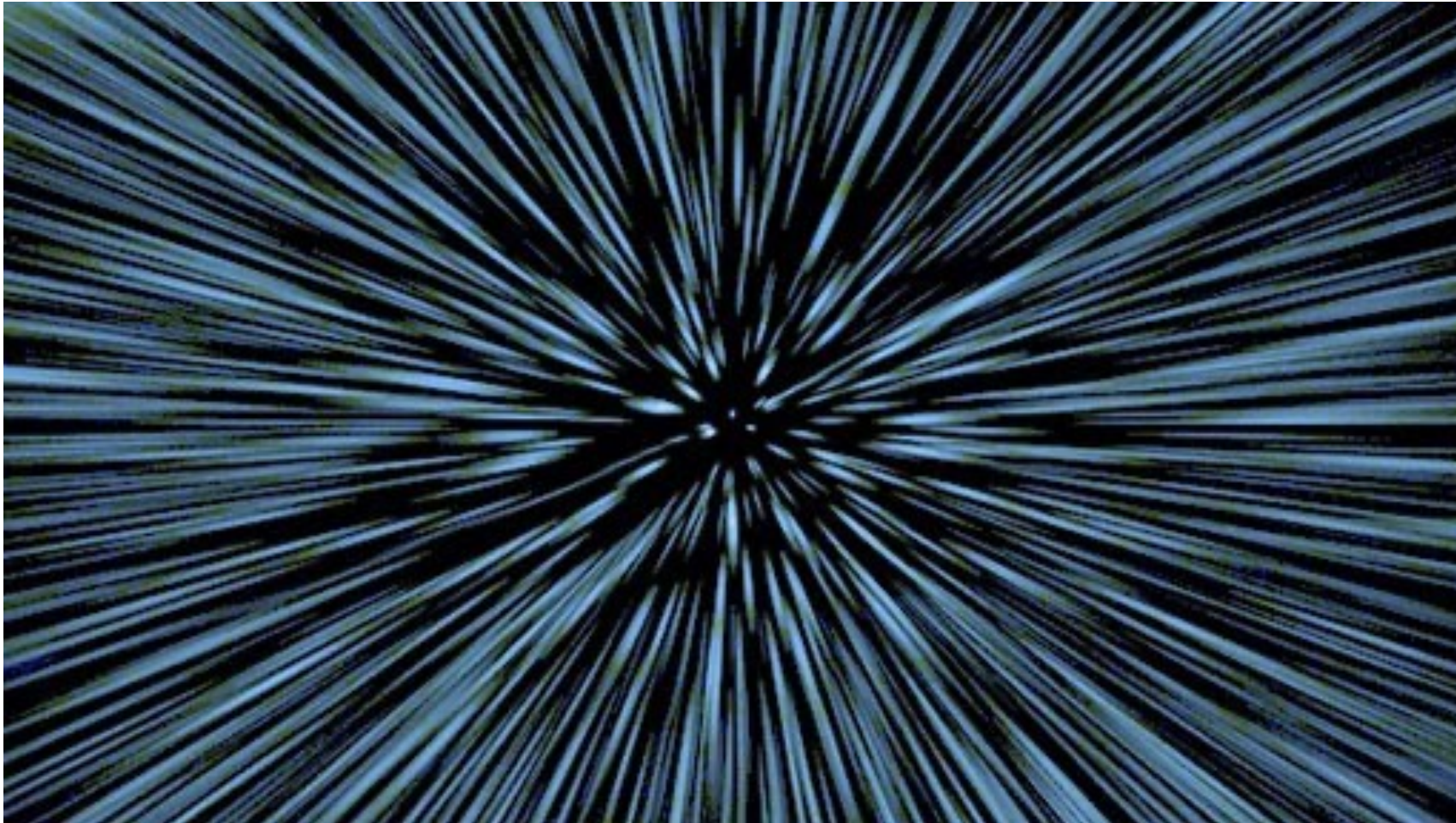# 2D transformations (a.k.a. warping)

Overview of today's lecture

- Reminder: image transformations.

- 2D transformations.

- Projective geometry 101.

- Transformations in projective geometry.

- Classification of 2D transformations.

- Determining unknown 2D transformations.

- Determining unknown image warps.

Slide credits

Most of these slides were adapted from:

- Kris Kitani (16-385, Spring 2017).
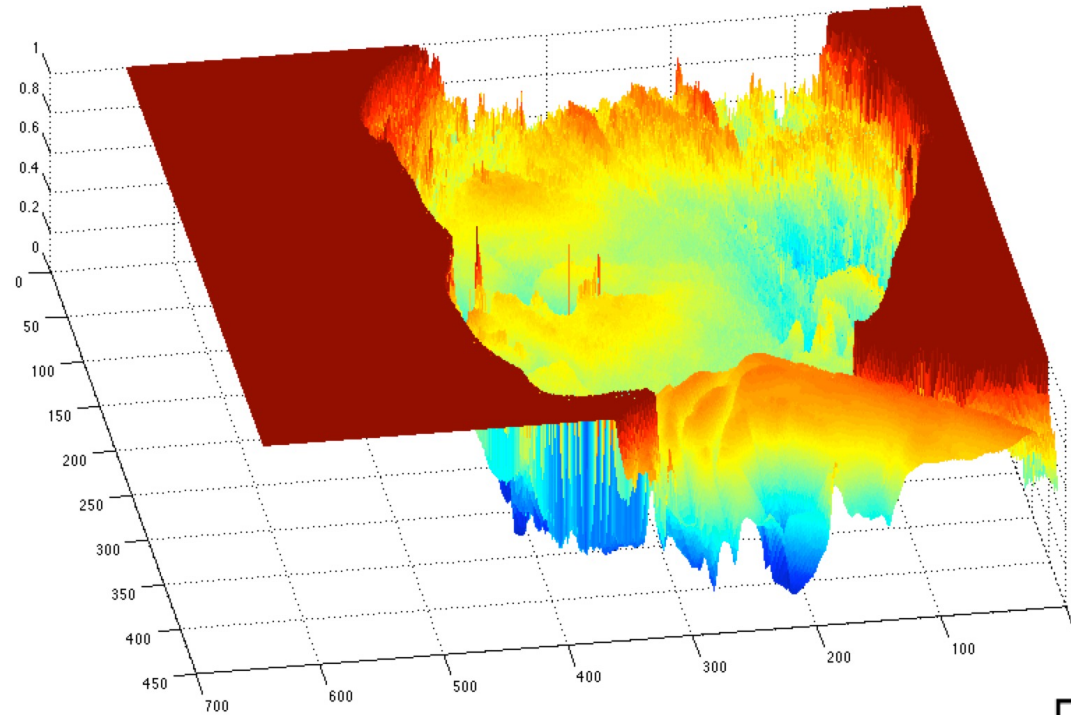
Reminder: image transformations

What is an image?

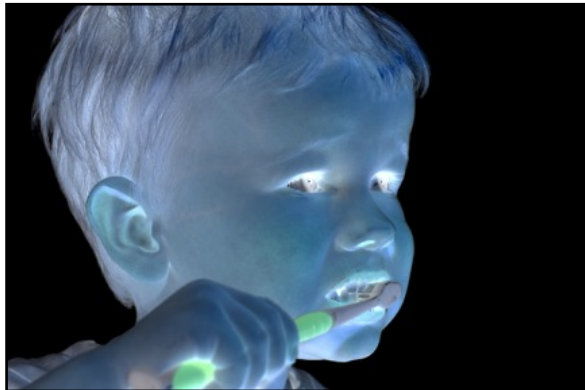grayscale image

What is the range of the image function f?

$f(\boldsymbol{x})$

A (grayscale) image is a 2D function.

domain $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

What types of image transformations can we do?
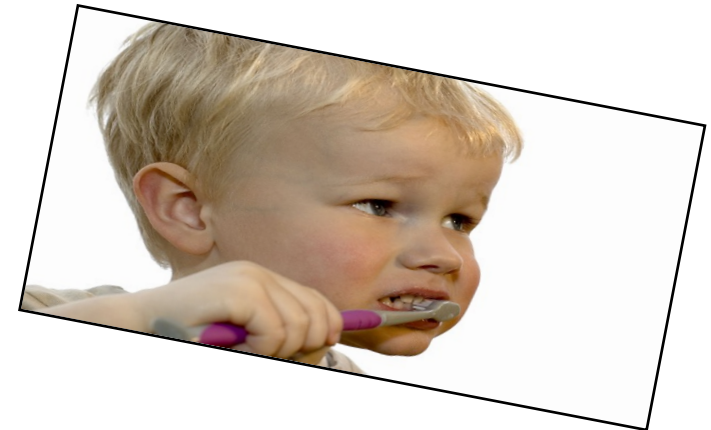

Filtering

changes pixel *values*

Warping

changes pixel *locations*

What types of image transformations can we do?

Filtering

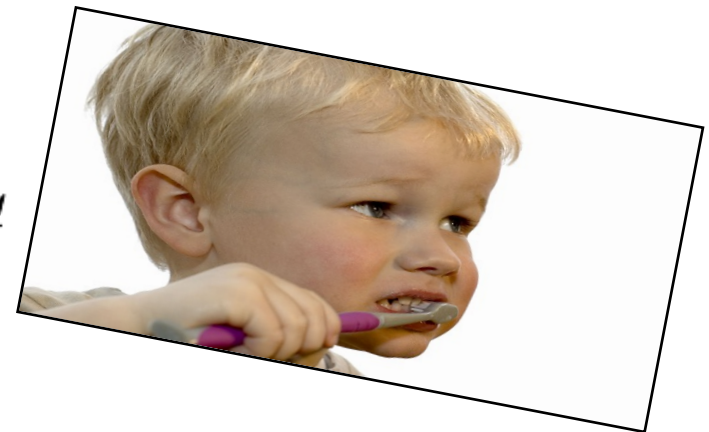$$G(\boldsymbol{x}) = h\{F(\boldsymbol{x})\}$$

changes *range* of image function

Warping

$$G(\boldsymbol{x}) = F(h\{\boldsymbol{x}\})$$
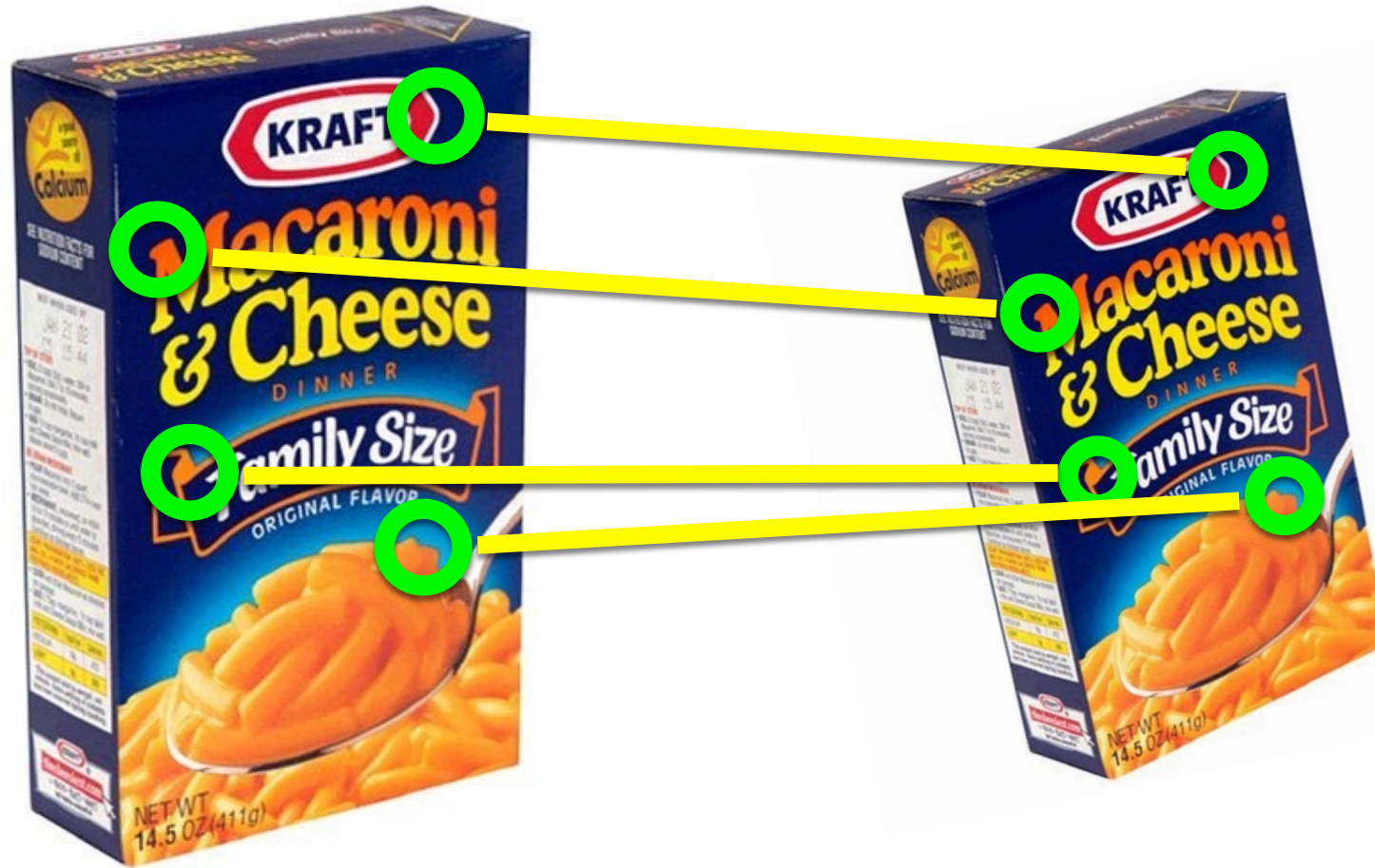
changes *domain* of image function

Warping example: feature matching

Warping example: feature matching

Warping example: feature matching

- object recognition
- 3D reconstruction
- augmented reality
- image stitching

How do you compute the transformation?

Warping example: feature matching

Given a set of matched feature points:

$$\{x_i, x_i'\}$$

point in one image
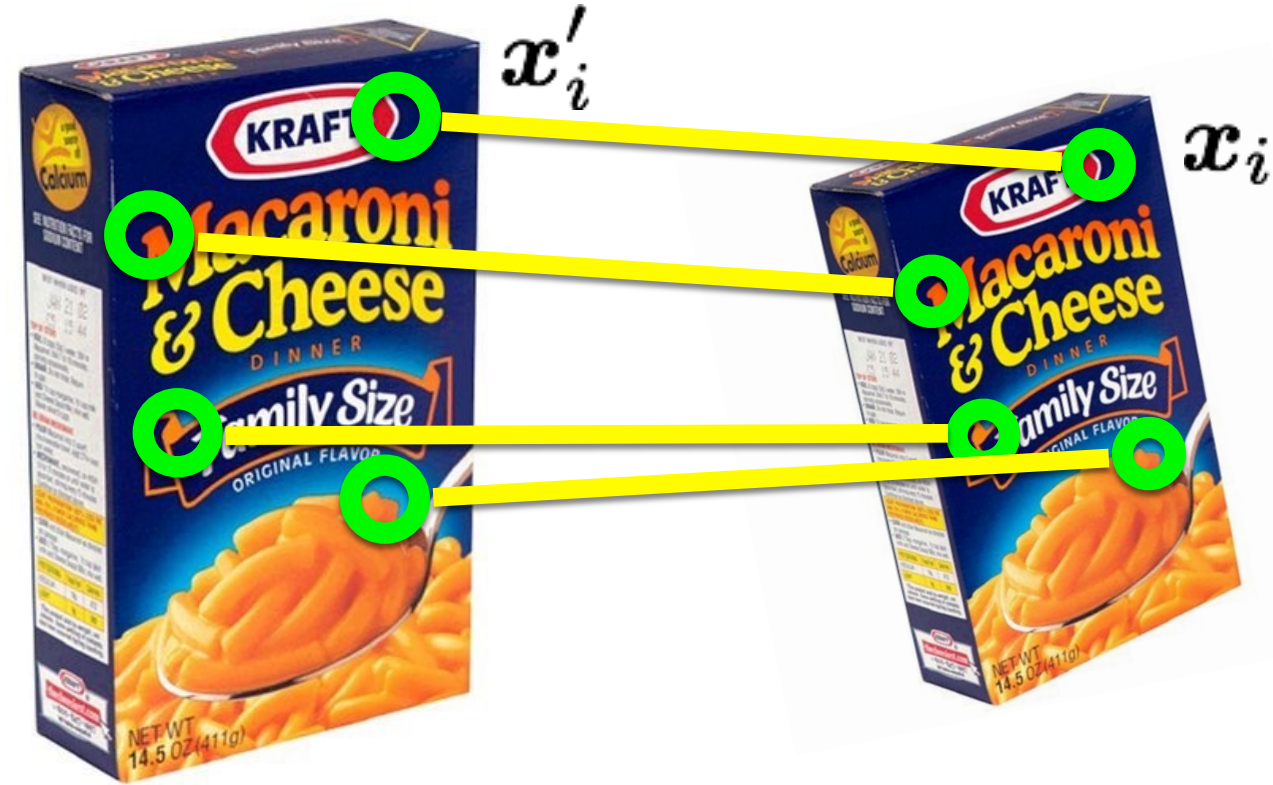
point in the other image

and a transformation:

$$x' = f(x; p)$$

transformation function

parameters

find the best estimate of the parameters

$$p$$

$$x_i'$$

$$x_i$$

What kind of transformation functions f are there?

2D transformations

2D transformations

translation

rotation

aspect

affine

perspective

cylindrical

2D planar transformations
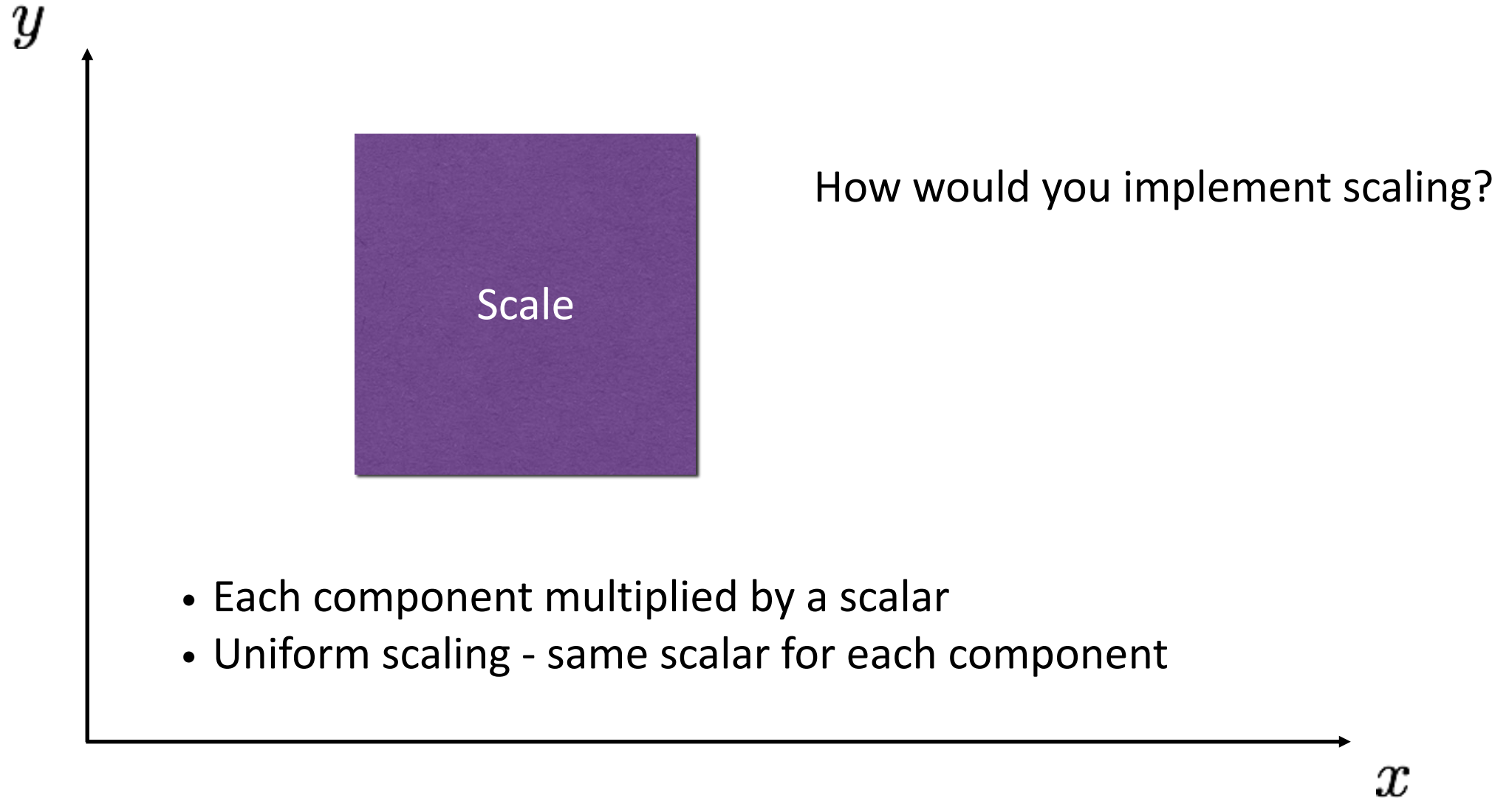
2D planar transformations

y

Scale

How would you implement scaling?

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component
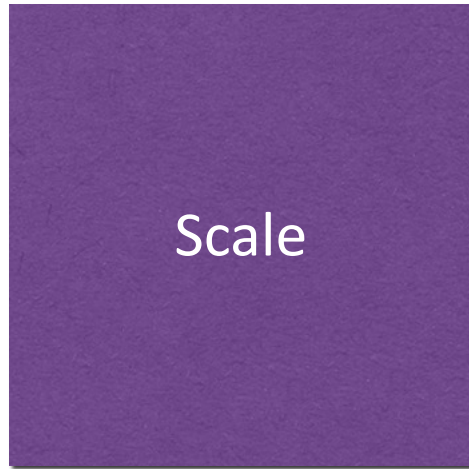
x

2D planar transformations

$$x' = ax$$

$$y' = by$$

What's the effect of using different scale factors?

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

2D planar transformations

$$x' = ax$$

$$y' = by$$

matrix representation of scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
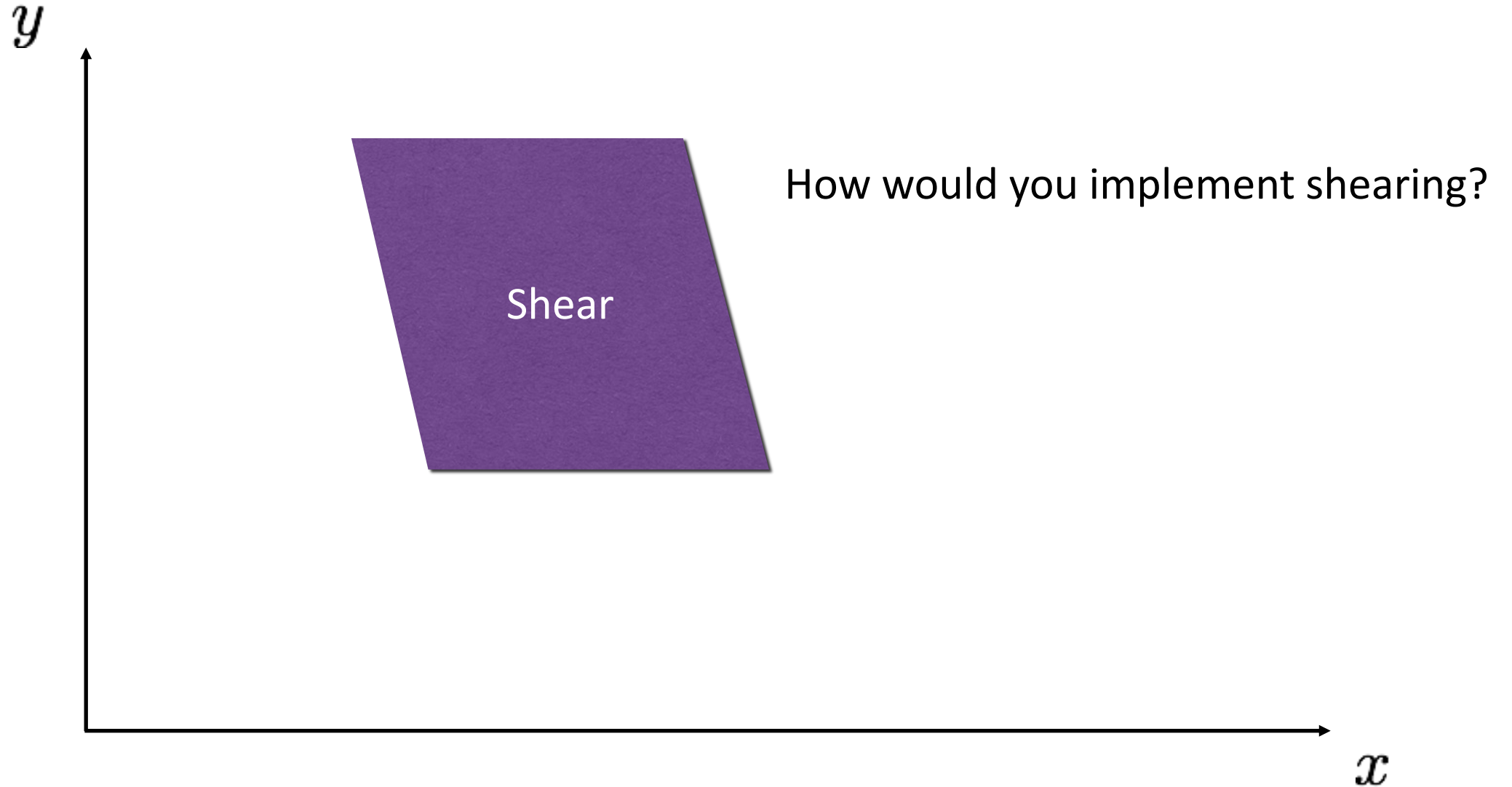
scaling matrix S

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

2D planar transformations

Shear

How would you implement shearing?

2D planar transformations

Shear

$$x' = x + a \cdot y$$

$$y' = b \cdot x + y$$

or in matrix form:

$$\left[\begin{array}{c} x' \\ y' \end{array} \right] = \left[\begin{array}{cc} 1 & a \\ b & 1 \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right]$$

2D planar transformations

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

rotation around
the origin

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

How would you implement rotation?

2D planar transformations

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$

rotation around
the origin

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

2D planar transformations

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

rotation around
the origin

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Polar coordinates...

x = r cos (φ)

y = r sin (φ)

x' = r cos (φ + θ)

y' = r sin (φ + θ)

Trigonometric Identity...

x' = r cos(φ) cos(θ) − r sin(φ) sin(θ)

y' = r sin(φ) cos(θ) + r cos(φ) sin(θ)

Substitute...

x' = x cos(θ) - y sin(θ)

y' = x sin(θ) + y cos(θ)

2D planar transformations

$$x' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

rotation around
the origin

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
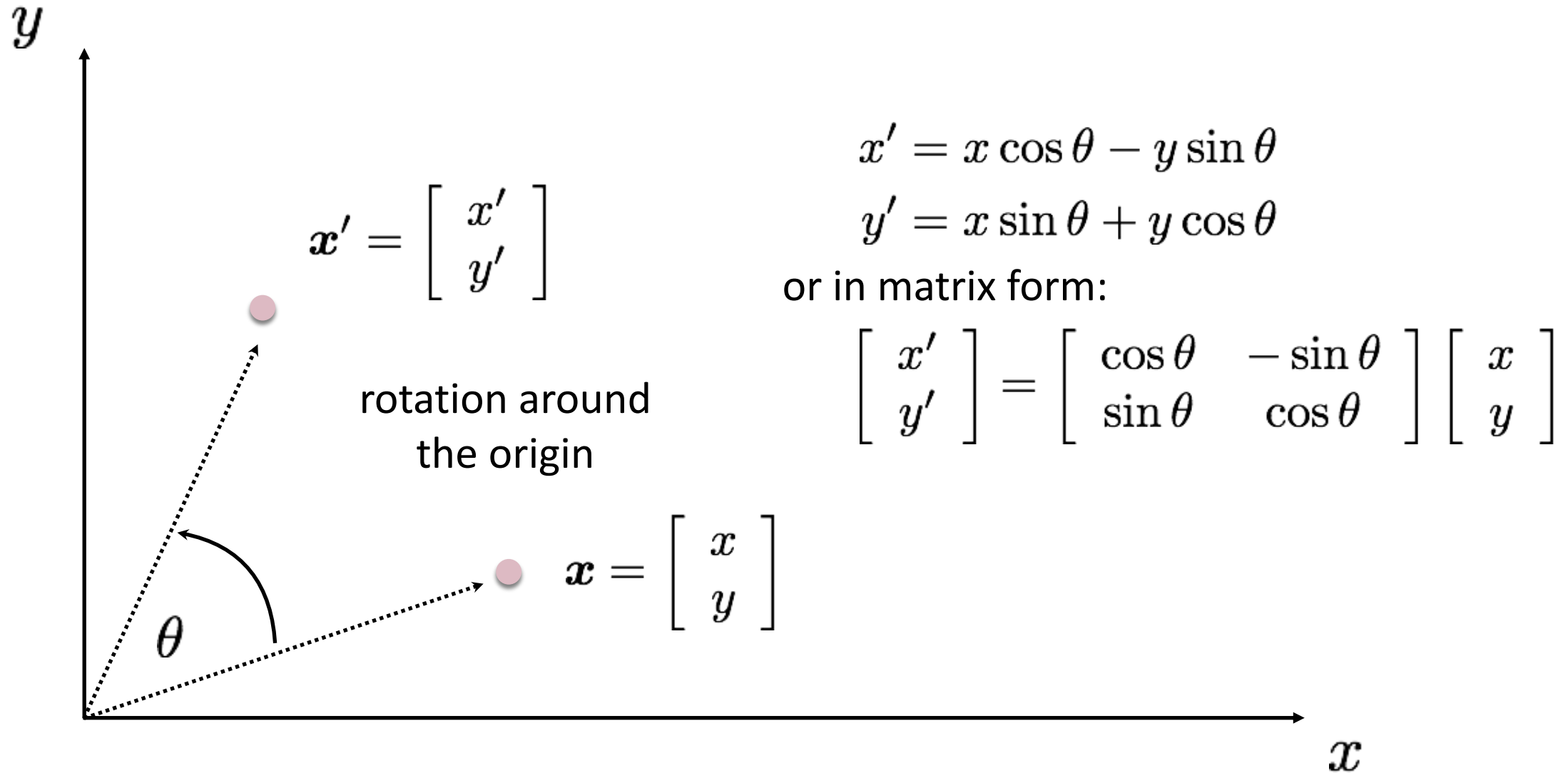
2D planar and linear transformations

$$\boldsymbol{x}' = f(\boldsymbol{x}; p)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

parameters p point $\boldsymbol{x}$

2D planar and linear transformations

Scale

$$\mathbf{M} = \left[\begin{array}{cc} s_x & 0 \\ 0 & s_y \end{array} \right]$$

Flip across y

$$\mathbf{M} = \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right]$$

Rotate

$$\mathbf{M} = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$$

Flip across origin

$$\mathbf{M} = \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right]$$

Shear

$$\mathbf{M} = \left[\begin{array}{cc} 1 & s_x \\ s_y & 1 \end{array} \right]$$

Identity

$$\mathbf{M} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$
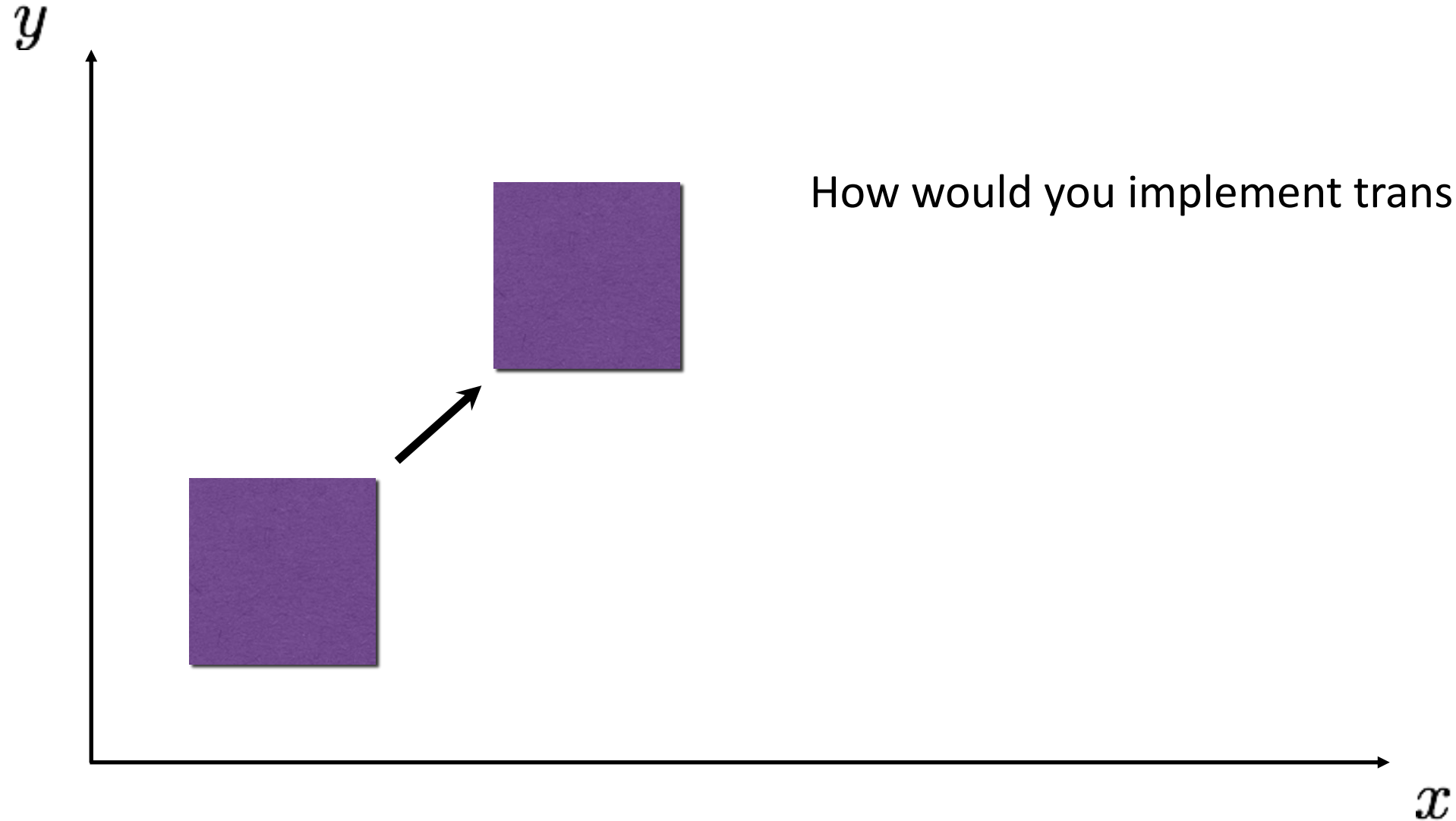
2D translation

How would you implement translation?

2D translation

$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation?

$$\mathbf{M} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

2D translation

$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation?

Not possible.
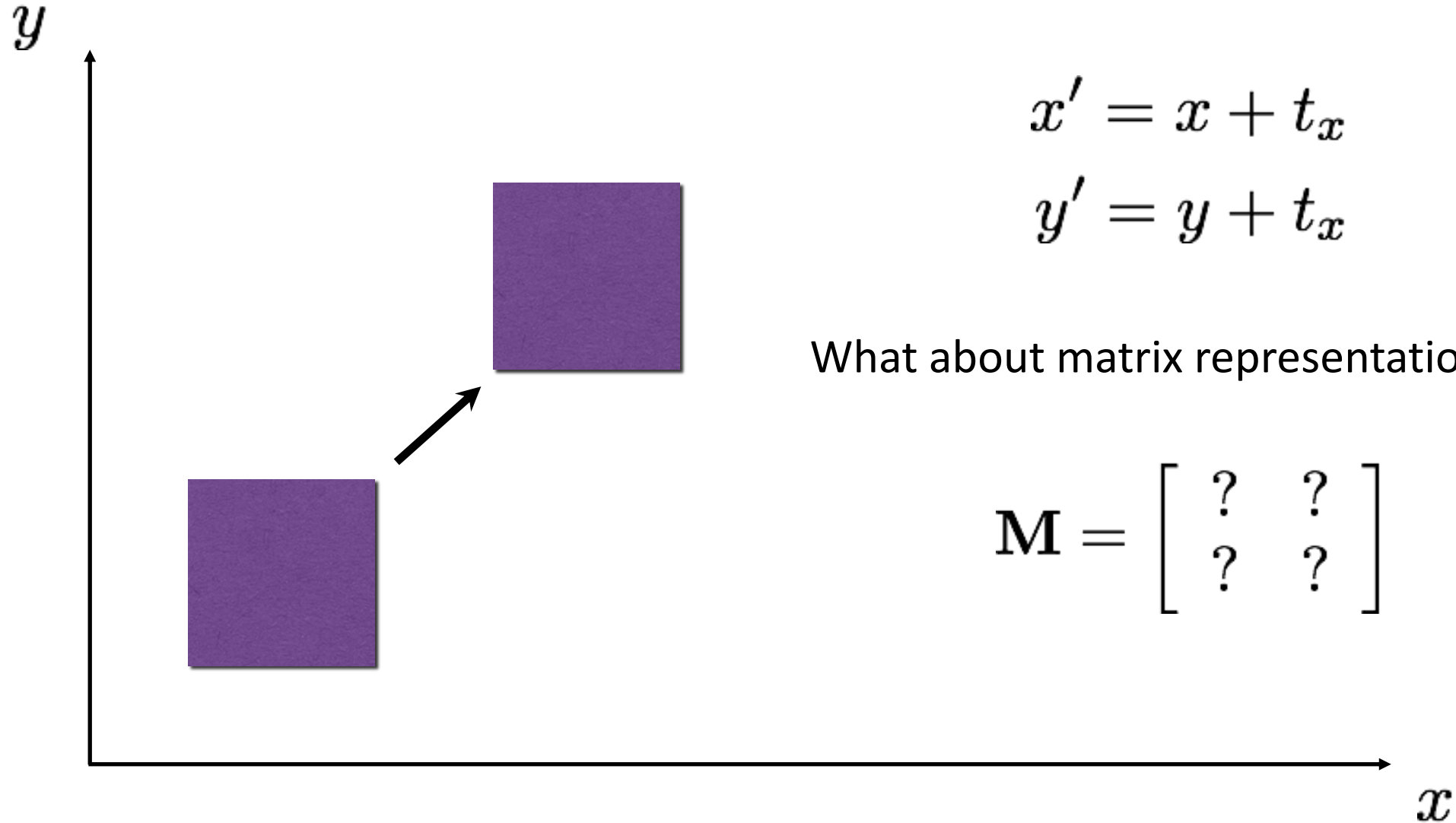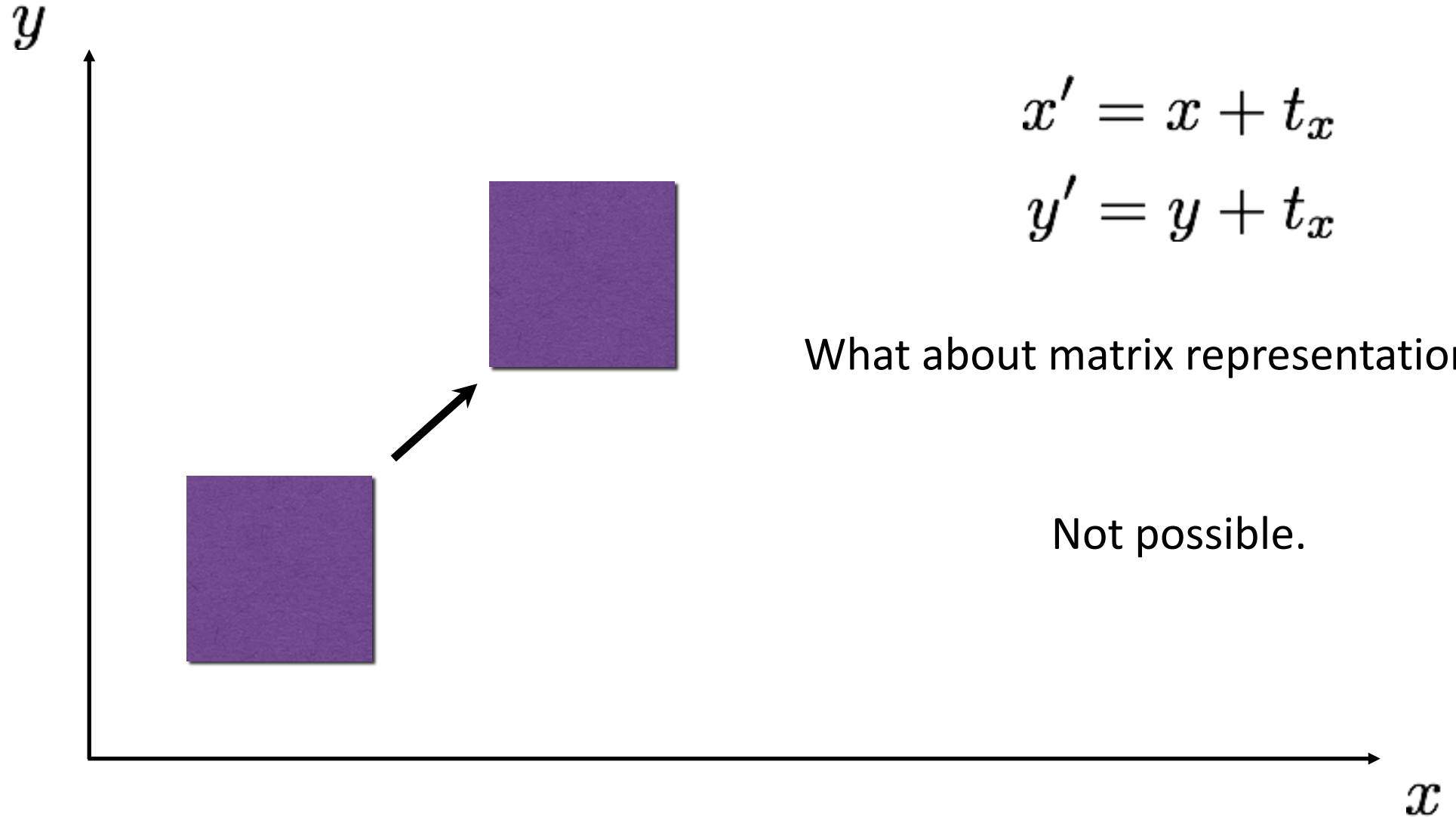
Projective geometry 101

Homogeneous coordinates

heterogeneous
coordinates

homogeneous
coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

add a 1 here

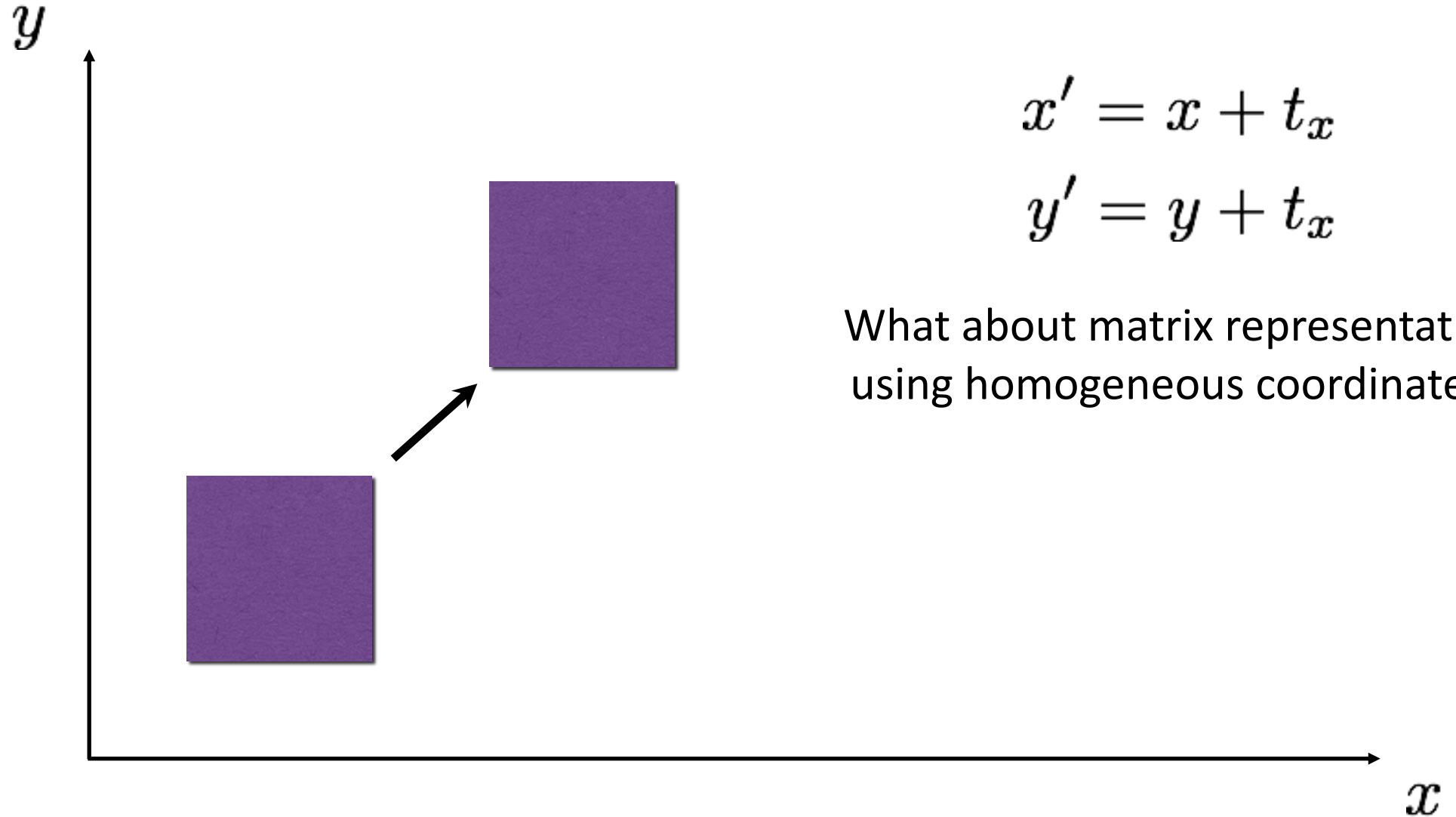- Represent 2D point with a 3D vector

Homogeneous coordinates

heterogeneous
coordinates

homogeneous
coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$
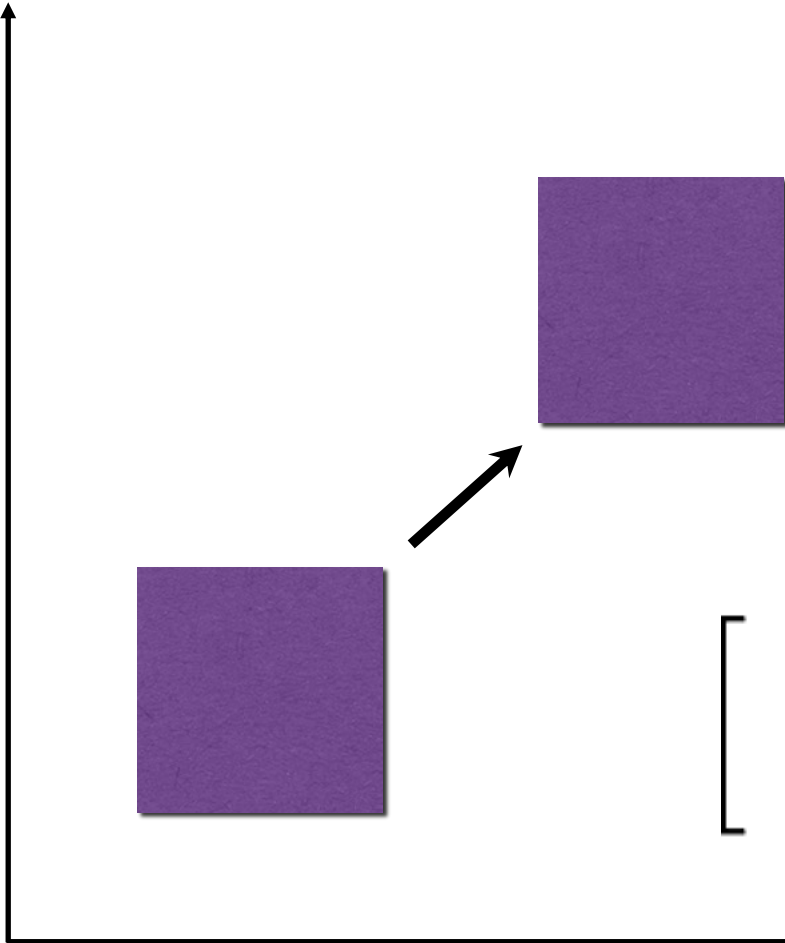
- Represent 2D point with a 3D vector
- 3D vectors are only defined up to scale

2D translation

$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation
using homogeneous coordinates?

2D translation

$$x' = x + t_x$$
$$y' = y + t_x$$
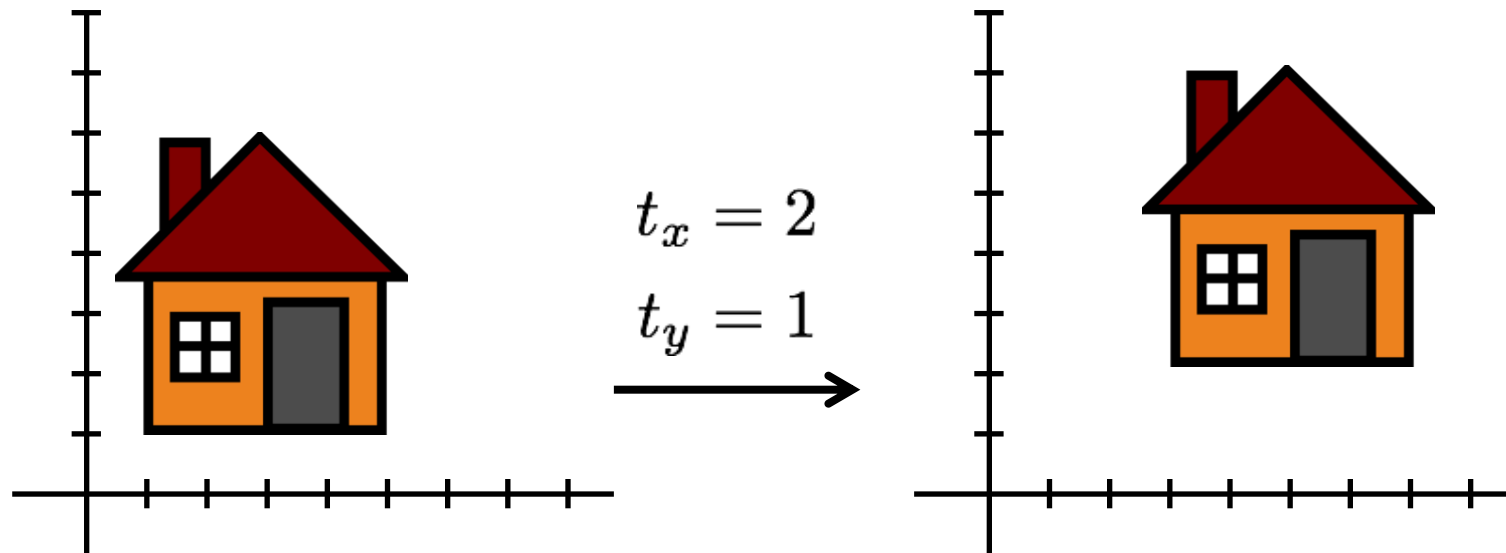
What about matrix representation
using homogeneous coordinates?

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

2D translation using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

$t_x = 2$

$t_y = 1$

Homogeneous coordinates

Conversion:

- heterogeneous → homogeneous

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- homogeneous → heterogeneous

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

- scale invariance

$$\begin{bmatrix} x & y & w \end{bmatrix}^\top = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^\top$$

Special points:

- point at infinity

$$\begin{bmatrix} x & y & 0 \end{bmatrix}$$

- undefined

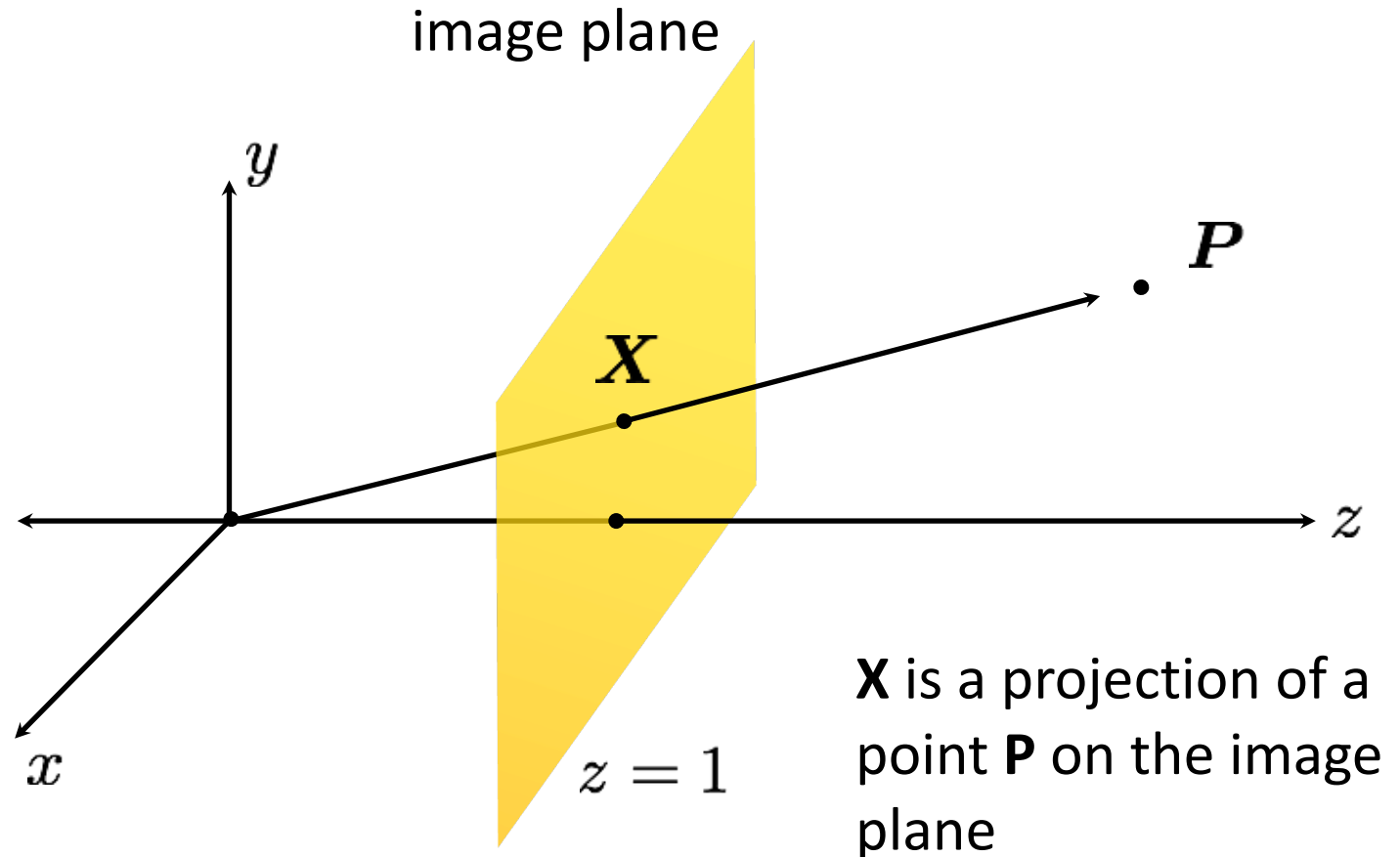$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Projective geometry

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Updownarrow$

image point in homogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

X is a projection of a point **P** on the image plane

Transformations in projective geometry

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p′   =   ?   ?   ?   **p**

Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p'   =   translation(t_x,t_y)   rotation(θ)   scale(s,s)   **p**

Does the multiplication order matter?

Classification of 2D transformations

Classification of 2D transformations
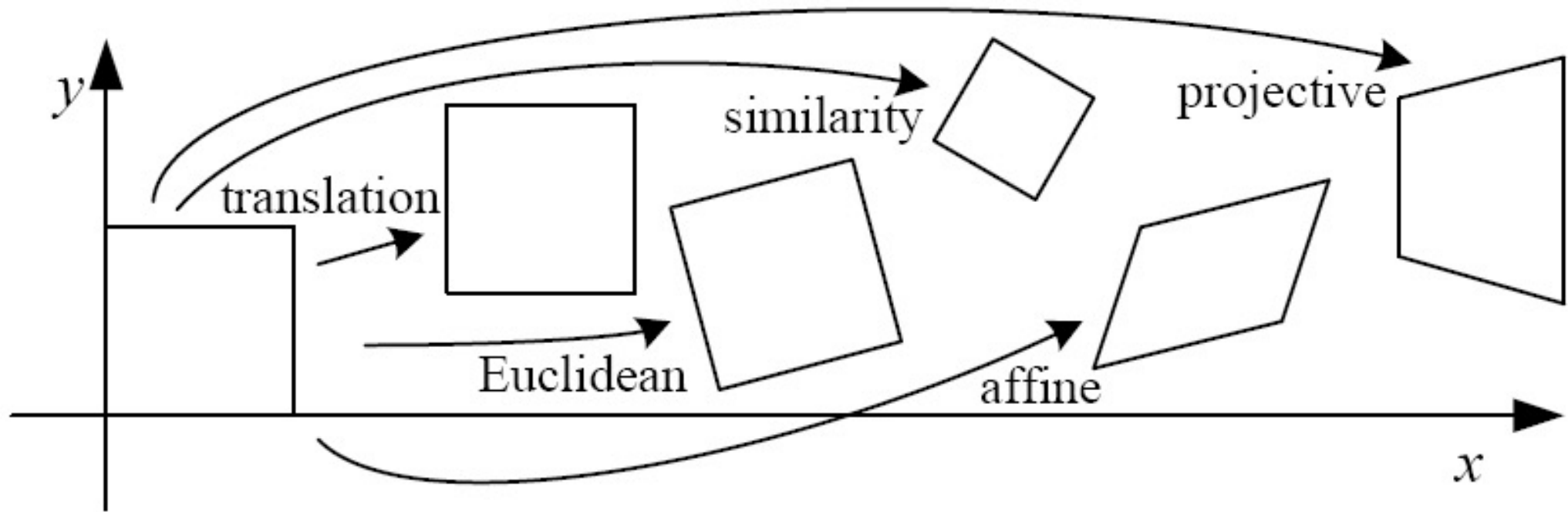
Classification of 2D transformations

Name	Matrix	# D.O.F.
translation	$\left[\begin{array}{c|c} I & t \end{array}\right]$	?
rigid (Euclidean)	$\left[\begin{array}{c|c} R & t \end{array}\right]$	?
similarity	$\left[\begin{array}{c|c} sR & t \end{array}\right]$	?
affine	$\left[\begin{array}{c} A \end{array}\right]$	?
projective	$\left[\begin{array}{c} \tilde{H} \end{array}\right]$	?

Classification of 2D transformations

Translation:

$$\begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?
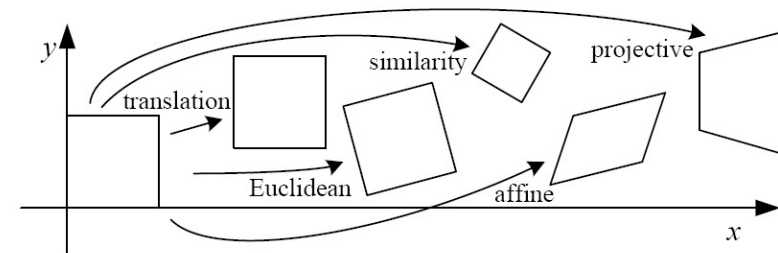
Classification of 2D transformations

Euclidean (rigid):
rotation + translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?

Classification of 2D transformations

Euclidean (rigid):
rotation + translation

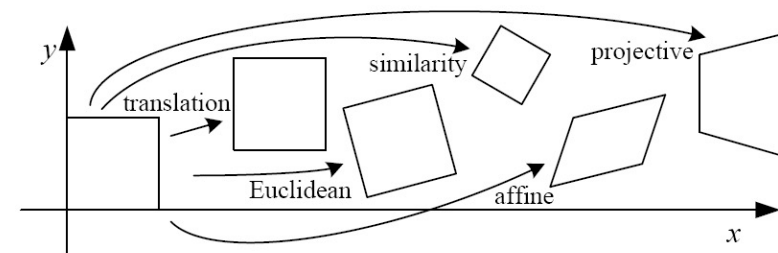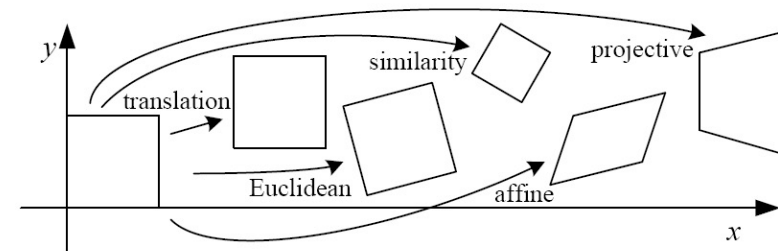$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?
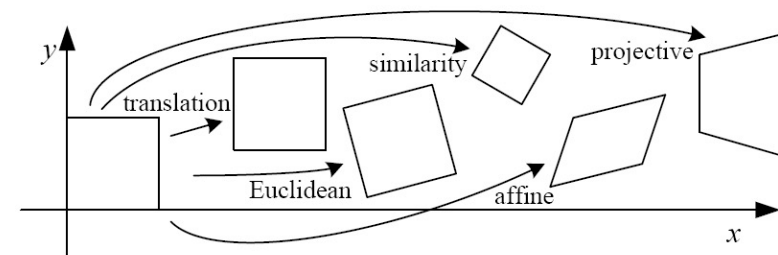
Classification of 2D transformations
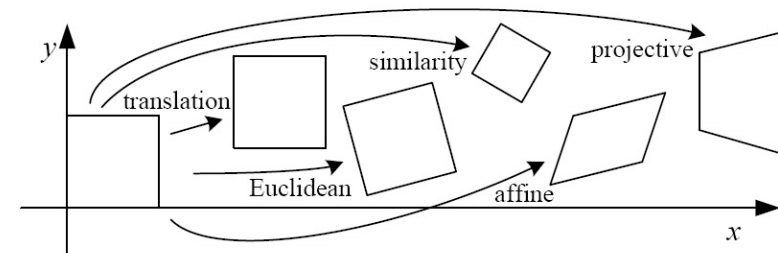
which other matrix values
will change if this increases?
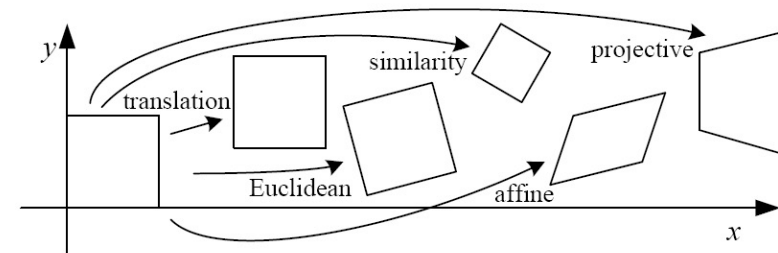
$$\downarrow$$

Euclidean (rigid):
rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Classification of 2D transformations

what will happen to the
image if this increases?

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Euclidean (rigid):
rotation + translation

Classification of 2D transformations

what will happen to the
image if this increases?

$$\downarrow$$
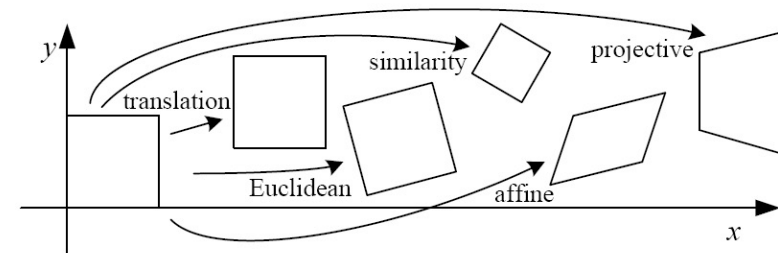
Euclidean (rigid):
rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

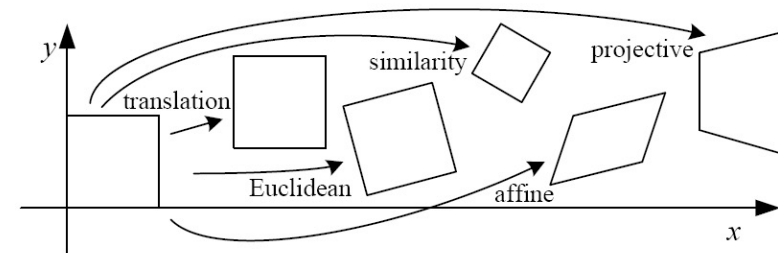# Classification of 2D transformations

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$
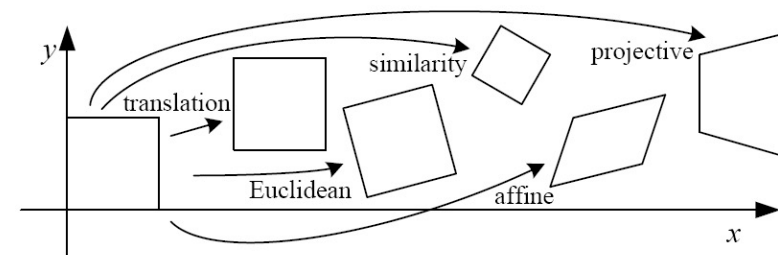
Are there any values that are related?

Classification of 2D transformations

multiply these four by scale **s**

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

Classification of 2D transformations

what will happen to the
image if this increases?

$$\downarrow$$

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$
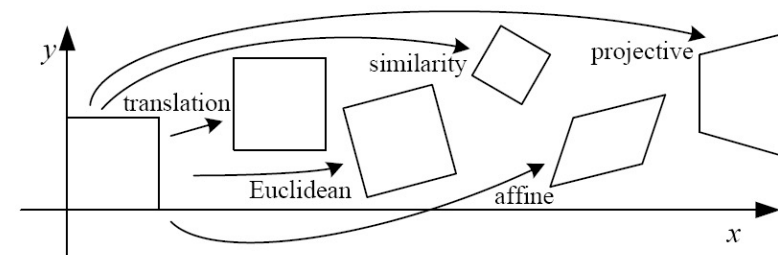
Classification of 2D transformations

Affine transform:
uniform scaling + shearing
+ rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?

Classification of 2D transformations

Affine transform:
uniform scaling + shearing
+ rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?

similarity   shear

$$\begin{bmatrix} sr_1 & sr_2 \\ sr_3 & sr_4 \end{bmatrix} \begin{bmatrix} 1 & h_1 \\ h_2 & 1 \end{bmatrix} = \begin{bmatrix} sr_1 + h_2 sr_2 & sr_2 + h_1 sr_1 \\ sr_3 + h_2 sr_4 & sr_4 + h_1 sr_3 \end{bmatrix}$$

# Classification of 2D transformations

Affine transform:
uniform scaling + shearing
+ rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

similarity   shear

$$\begin{bmatrix} sr_1 & sr_2 \\ sr_3 & sr_4 \end{bmatrix} \begin{bmatrix} 1 & h_1 \\ h_2 & 1 \end{bmatrix} = \begin{bmatrix} sr_1 + h_2 sr_2 & sr_2 + h_1 sr_1 \\ sr_3 + h_2 sr_4 & sr_4 + h_1 sr_3 \end{bmatrix}$$
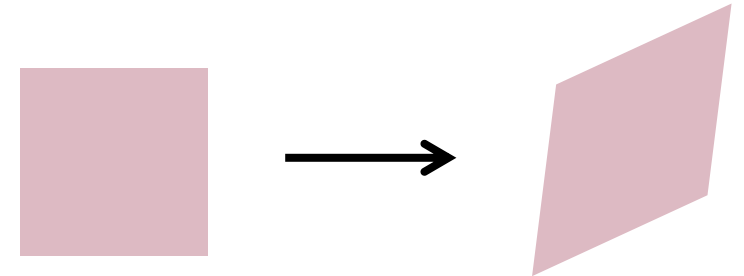
Affine transformations

Affine transformations are combinations of

- arbitrary (4-DOF) linear transformations; and

- translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- origin does not necessarily map to origin

- lines map to lines

- parallel lines map to parallel lines

- ratios are preserved

- compositions of affine transforms are also affine transforms
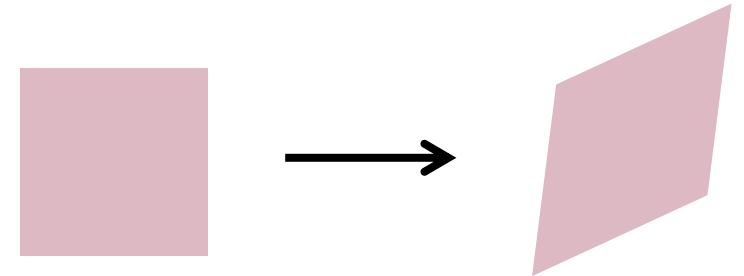
Does the last coordinate w ever change?

Affine transformations

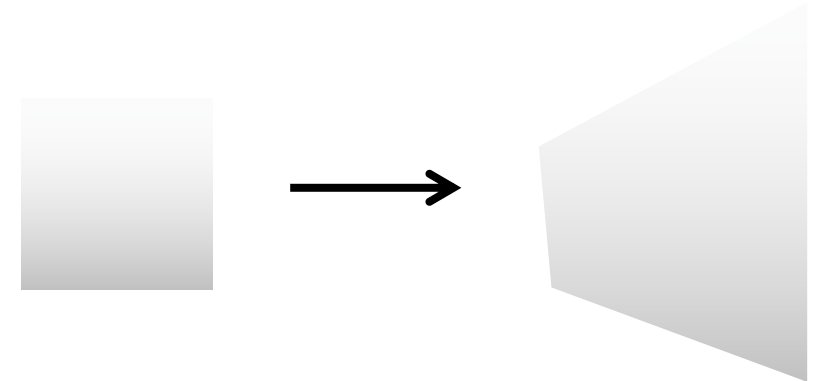Affine transformations are combinations of

- arbitrary (4-DOF) linear transformations; and

- translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- origin does not necessarily map to origin

- lines map to lines

- parallel lines map to parallel lines

- ratios are preserved

- compositions of affine transforms are also affine transforms

Nope! But what does that mean?

How to interpret affine transformations here?

image point in pixel coordinates

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

image point in heterogeneous coordinates

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

image plane

y

$\boldsymbol{P}$

$\boldsymbol{X}$

z

x

$z = 1$

X is a projection of a point **P** on the image plane

Projective transformations (aka homographies)
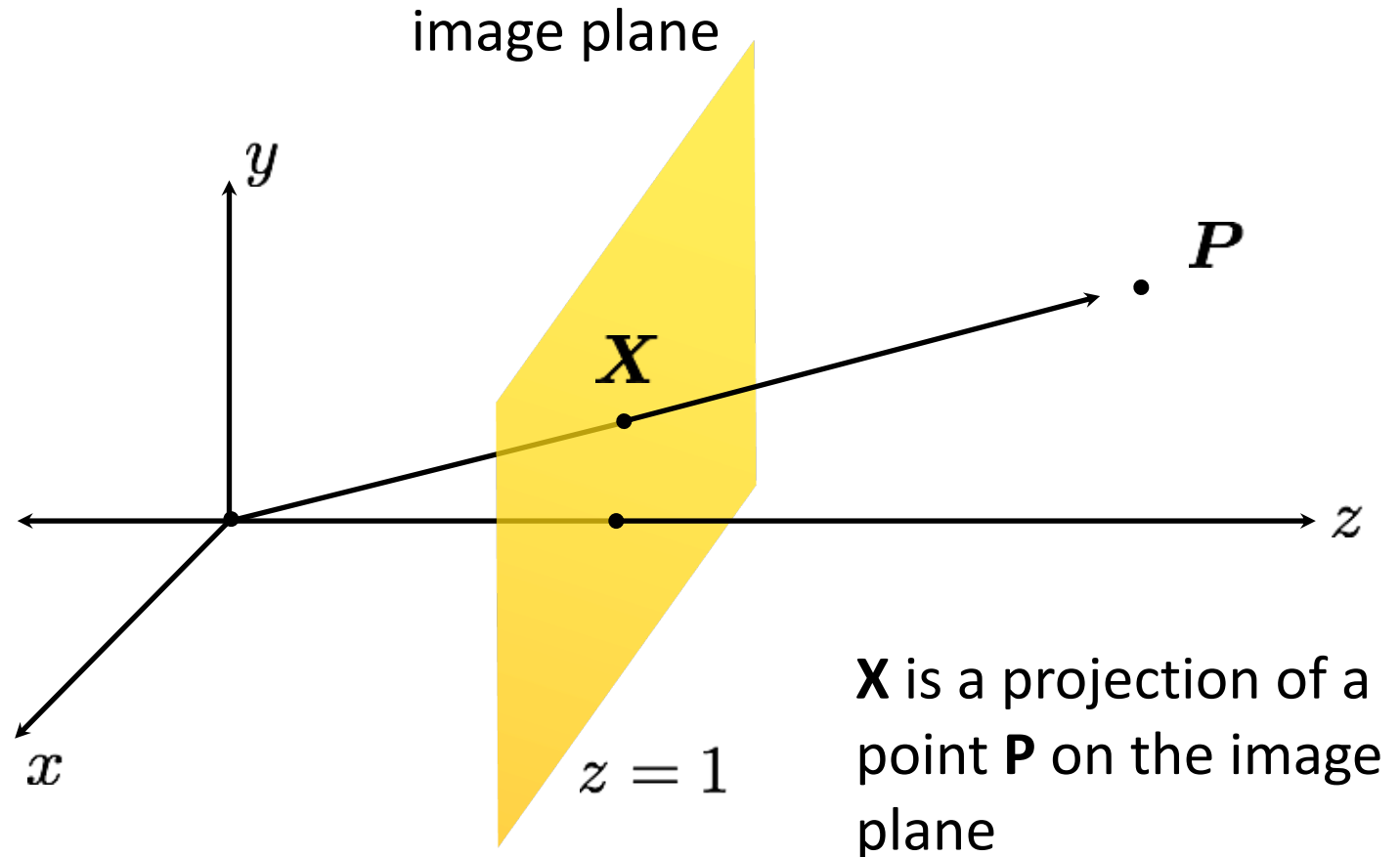
Projective transformations are combinations of

- affine transformations; and
- projective wraps

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms
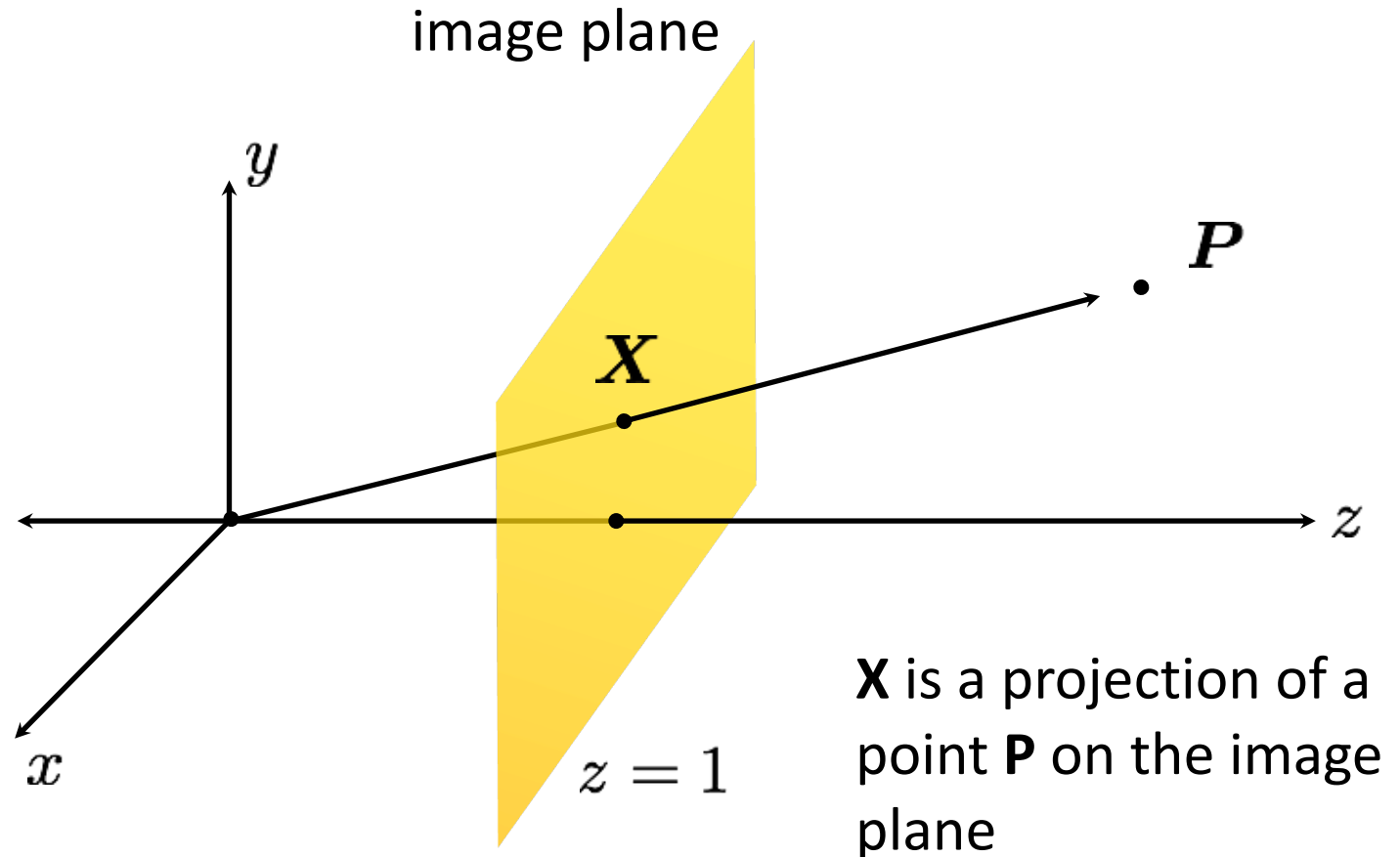
$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

How many degrees of freedom?

Projective transformations (aka homographies)

Projective transformations are combinations of

- affine transformations; and

- projective wraps

Properties of projective transformations:

- origin does not necessarily map to origin

- lines map to lines

- parallel lines do not necessarily map to parallel lines

- ratios are not necessarily preserved

- compositions of projective transforms are also projective transforms

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

8 DOF: vectors (and therefore matrices) are defined up to scale)
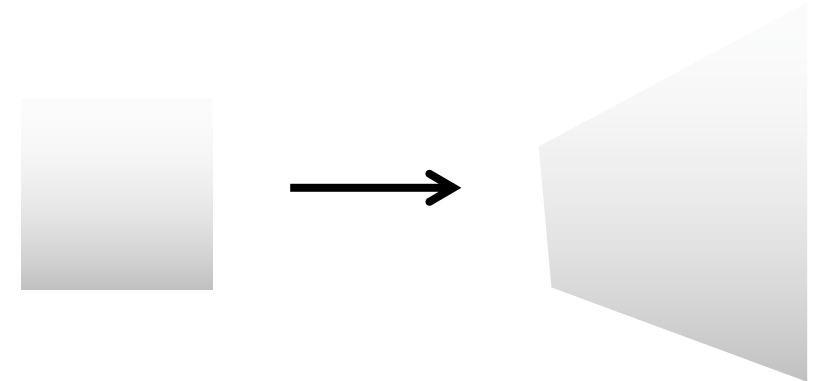
How to interpret projective transformations here?

image point in pixel coordinates

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Updownarrow$

image point in heterogeneous coordinates

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

image plane

y

$\boldsymbol{P}$

$\boldsymbol{X}$

z

x

$z = 1$

X is a projection of a point **P** on the image plane

Determining unknown (affine) 2D transformations

Determining unknown transformations

Suppose we have two triangles: ABC and DEF.
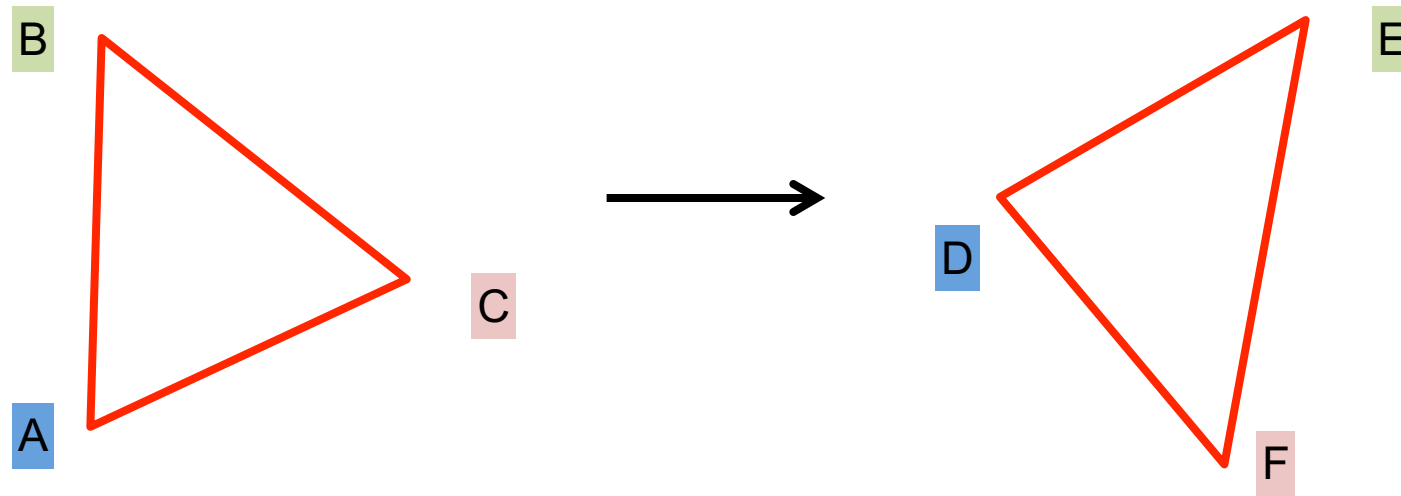
Determining unknown transformations

Suppose we have two triangles: ABC and DEF.
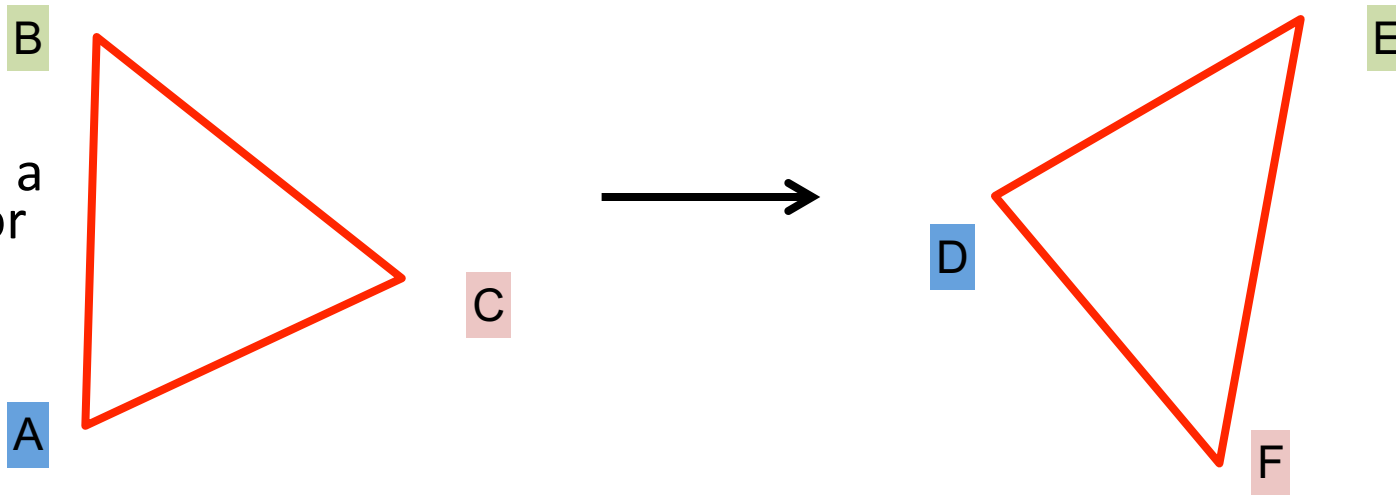- What type of transformation will map A to D, B to E, and C to F?
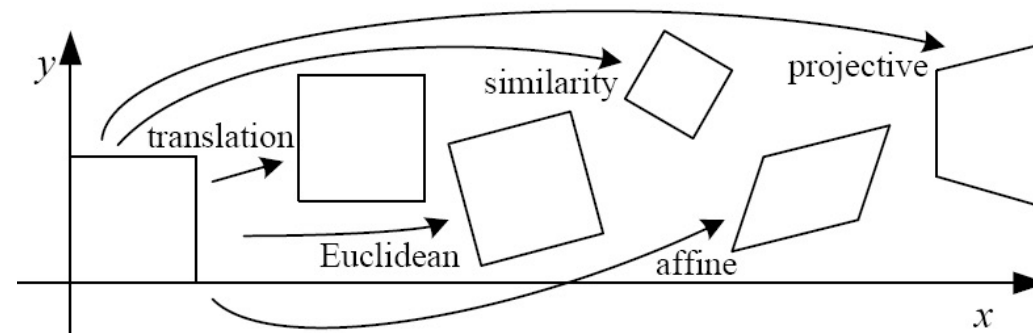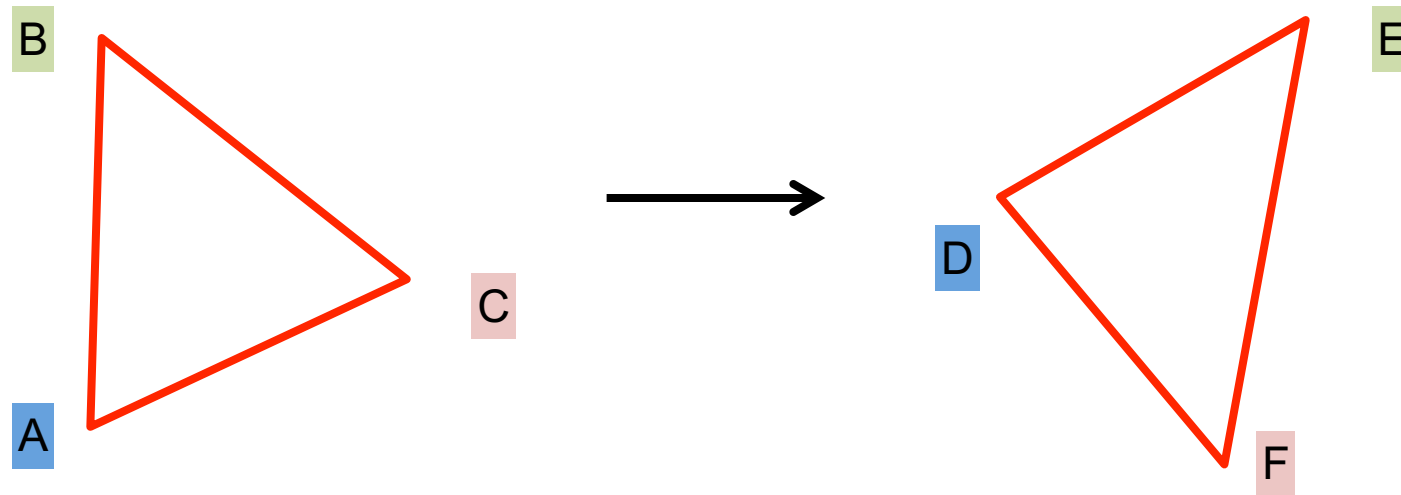
Determining unknown transformations

Suppose we have two triangles: ABC and DEF.
- What type of transformation will map A to D, B to E, and C to F?
- How do we determine the unknown parameters?

Important: We will see a different procedure for dealing with homographies!

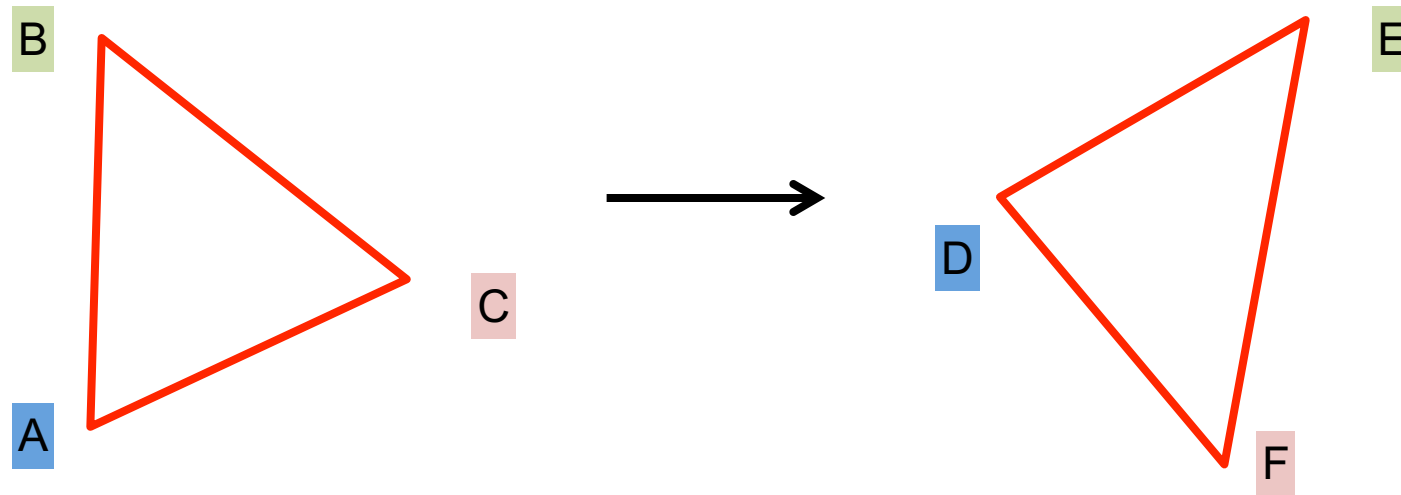Affine transform: uniform scaling + shearing + rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom do we have?

Determining unknown transformations

Suppose we have two triangles: ABC and DEF.
- What type of transformation will map A to D, B to E, and C to F?
- How do we determine the unknown parameters?
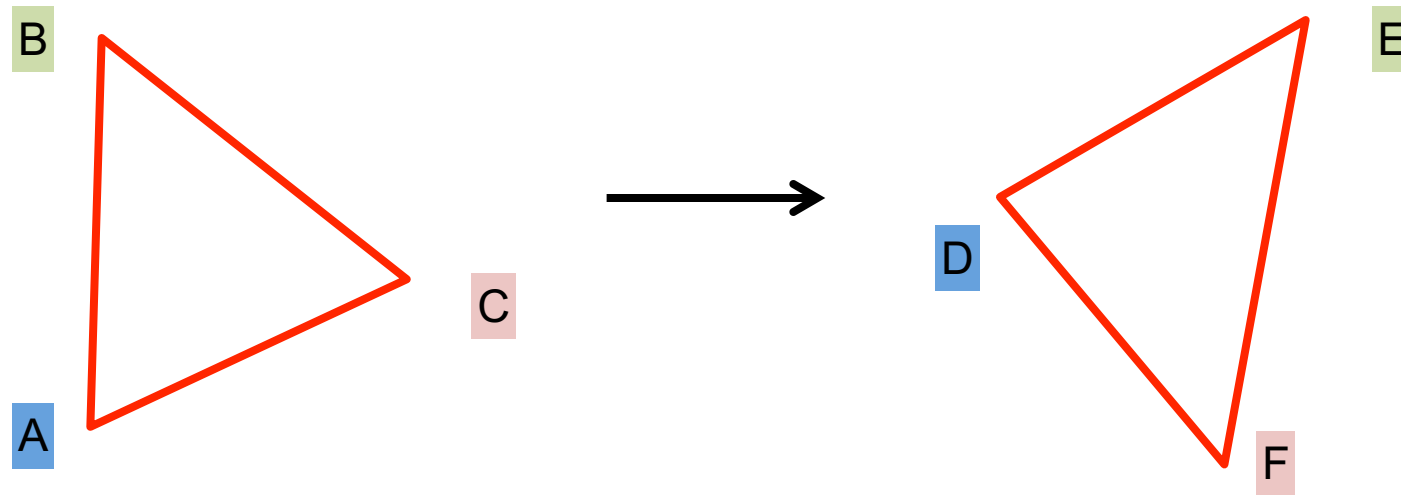

unknowns

$$x' = \mathbf{M}x$$

point correspondences

- One point correspondence gives how many equations?
- How many point correspondences do we need?

Determining unknown transformations

Suppose we have two triangles: ABC and DEF.
- What type of transformation will map A to D, B to E, and C to F?
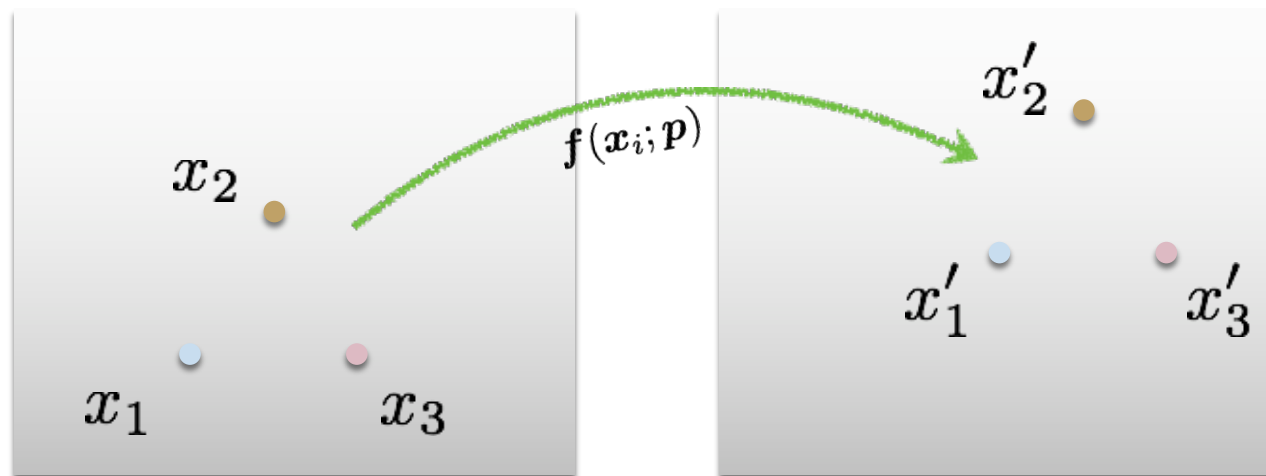- How do we determine the unknown parameters?

unknowns

$$x' = \mathbf{M}x$$

point correspondences

How do we solve this for **M**?
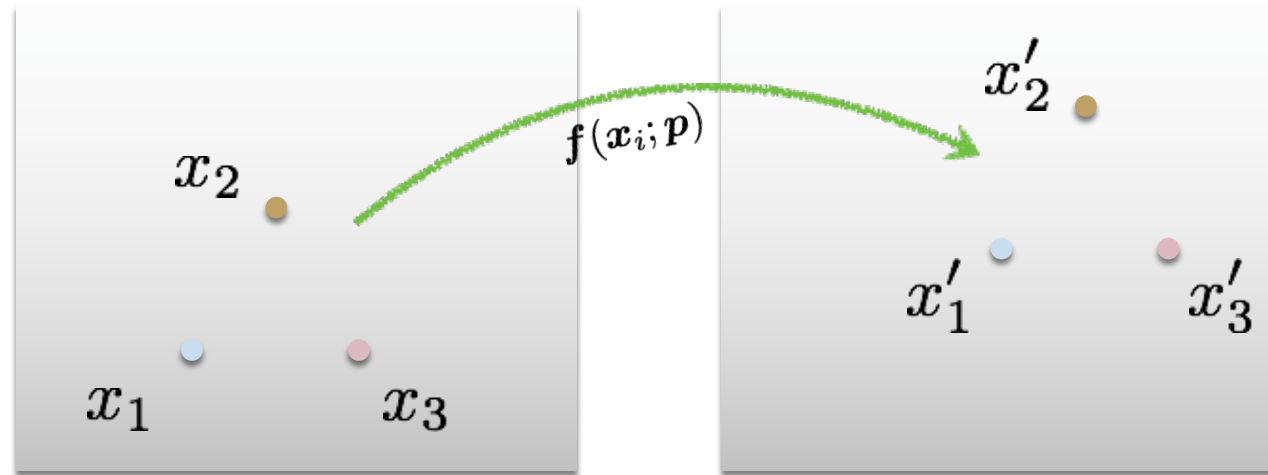
Least Squares Error

$$E_{\text{LS}} = \sum_i \| f(x_i; p) - x'_i \|^2$$

Least Squares Error
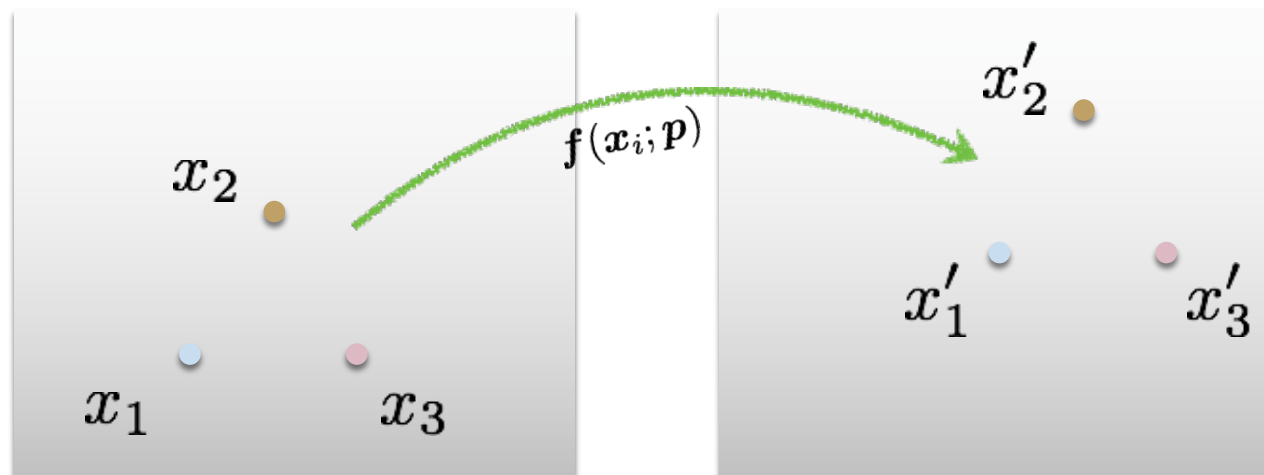
What is this?

What is this?

What is this?

$$E_{\mathrm{LS}} = \sum_i \| f(x_i; p) - x_i' \|^2$$

Least Squares Error

$$||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Euclidean (L2) norm

squared!

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}_i'\|^2$$

predicted location

measured location
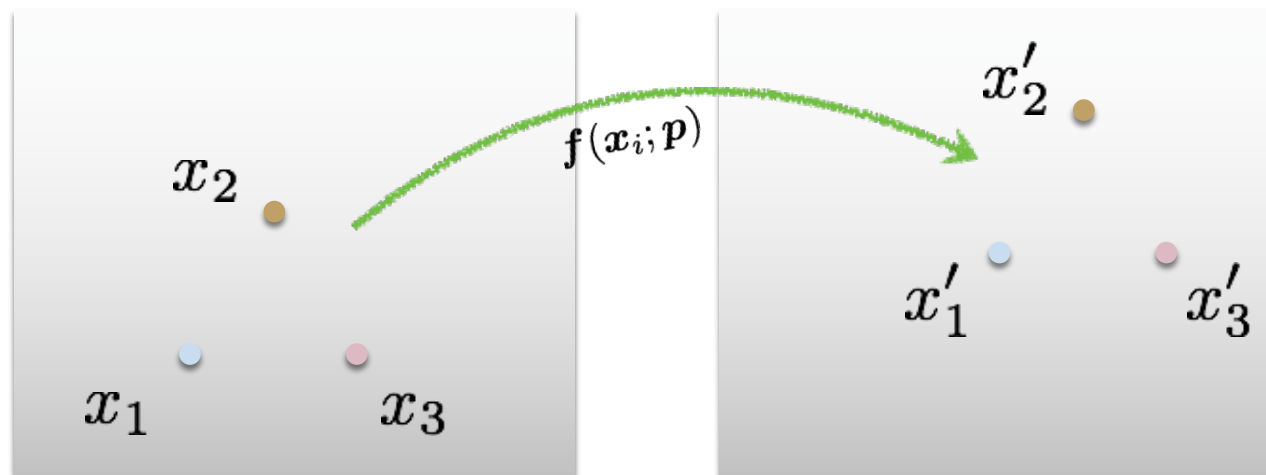
Least Squares Error

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

Residual (projection error)
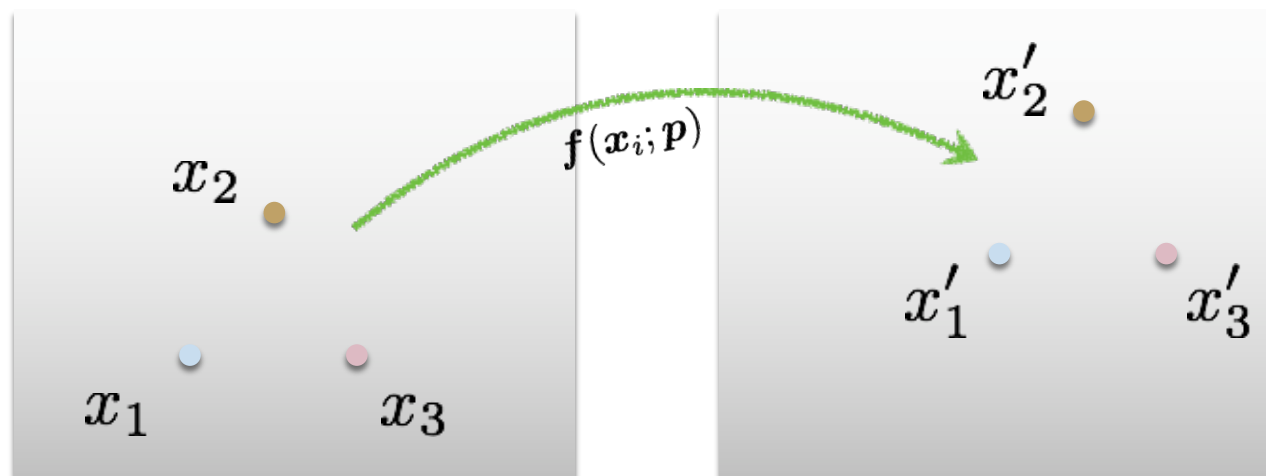
Least Squares Error

$$E_{\mathrm{LS}} = \sum_i \| f(x_i; p) - x'_i \|^2$$

What do we want to optimize?
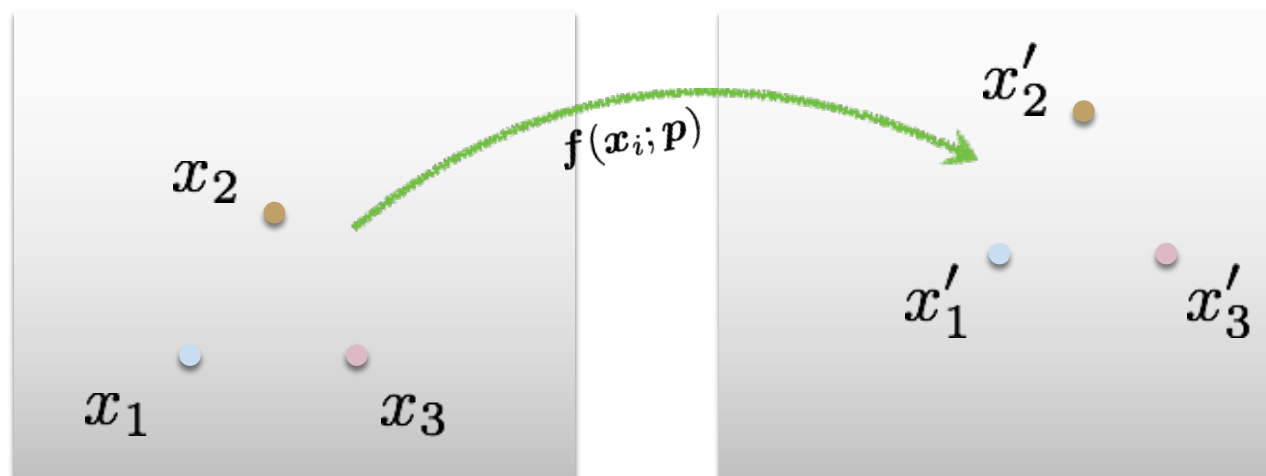
Find parameters that minimize squared error

$$\hat{\boldsymbol{p}} = \arg\min_{\boldsymbol{p}} \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}_i'\|^2$$

General form of linear least squares

$$E_{\text{LLS}} = \sum_i |\boldsymbol{a}_i \boldsymbol{x} - \boldsymbol{b}_i|^2$$

$$= \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|^2 \qquad \text{(matrix form)}$$

Determining unknown transformations

Affine transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Why can we drop the last line?

Vectorize transformation parameters:

Stack equations from point correspondences:

$$\begin{bmatrix} x' \\ y' \\ x' \\ y' \\ \vdots \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \\ x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \\ & & \vdots & & \\ x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix}$$

Notation in system form:
$$\underbrace{}_{b} \qquad \underbrace{}_{A} \qquad \underbrace{}_{x}$$

General form of linear least squares

$$E_{\text{LLS}} = \sum_i |a_i x - b_i|^2$$

$$= \|\mathbf{A}x - b\|^2 \quad \text{(matrix form)}$$

This function is quadratic.
How do you find the root of a quadratic?

Solving the linear system

Convert the system to a linear least-squares problem:

$$E_{\mathrm{LLS}} = \|\mathbf{A}x - b\|^2$$

Expand the error:

$$E_{\mathrm{LLS}} = x^\top(\mathbf{A}^\top\mathbf{A})x - 2x^\top(\mathbf{A}^\top b) + \|b\|^2$$

Minimize the error:

Set derivative to 0 $\quad (\mathbf{A}^\top\mathbf{A})x = \mathbf{A}^\top b$

Solve for x $\quad x = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top b \longleftarrow$

In Python:

```
x = numpy.linalg.
    solve(A, b)
```

Note: You almost <u>never</u> want to compute the inverse of a matrix.

Linear least squares estimation only works when the transform function is ?

Linear least squares estimation only works when the transform function is **linear! (duh)**

Also doesn't deal well with outliers