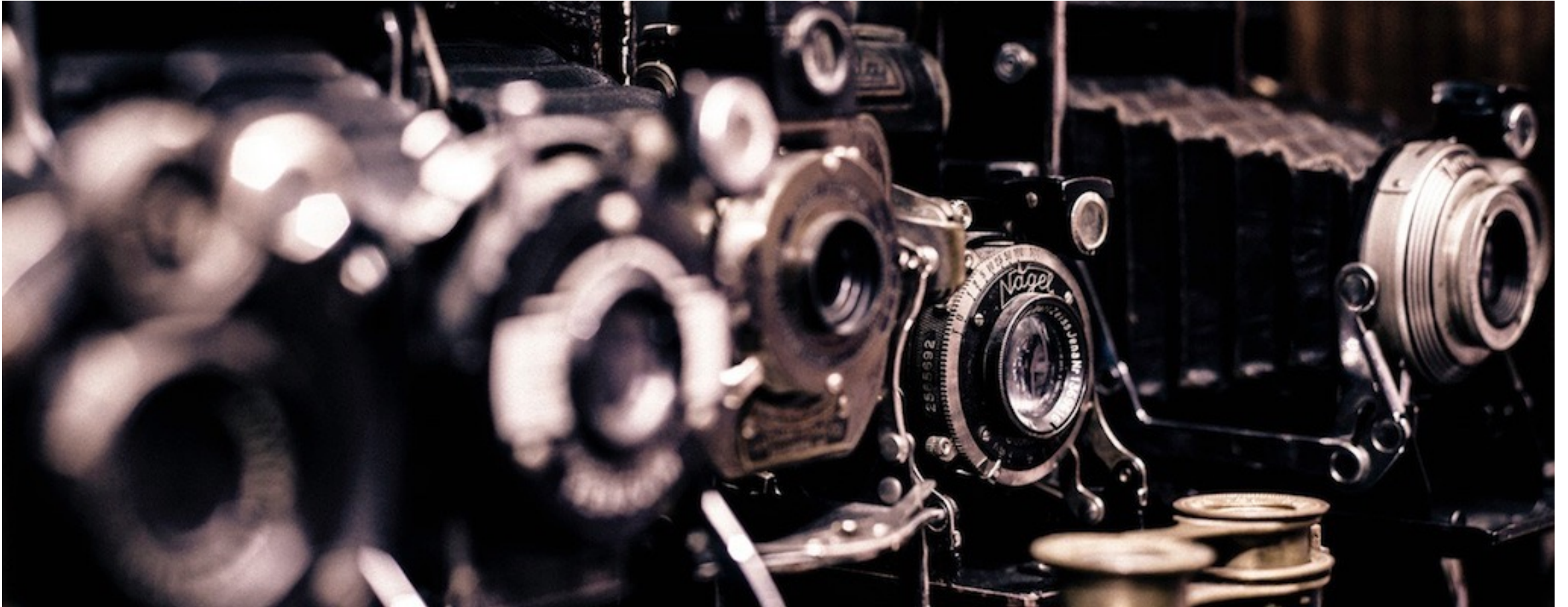


Geometric camera models



Overview of today's lecture

- Some motivational imaging experiments.
- Pinhole camera.
- Accidental pinholes.
- Camera matrix.

Slide credits

Most of these slides were adapted from:

- Kris Kitani (15-463, Fall 2016).

Some slides inspired from:

- Fredo Durand (MIT).

Some motivational imaging experiments

Let's say we have a sensor...



digital sensor
(CCD or CMOS)

... and an object we like to photograph

real-world
object



digital sensor
(CCD or CMOS)



What would an image taken like this look like?

Bare-sensor imaging

real-world
object



digital sensor
(CCD or CMOS)

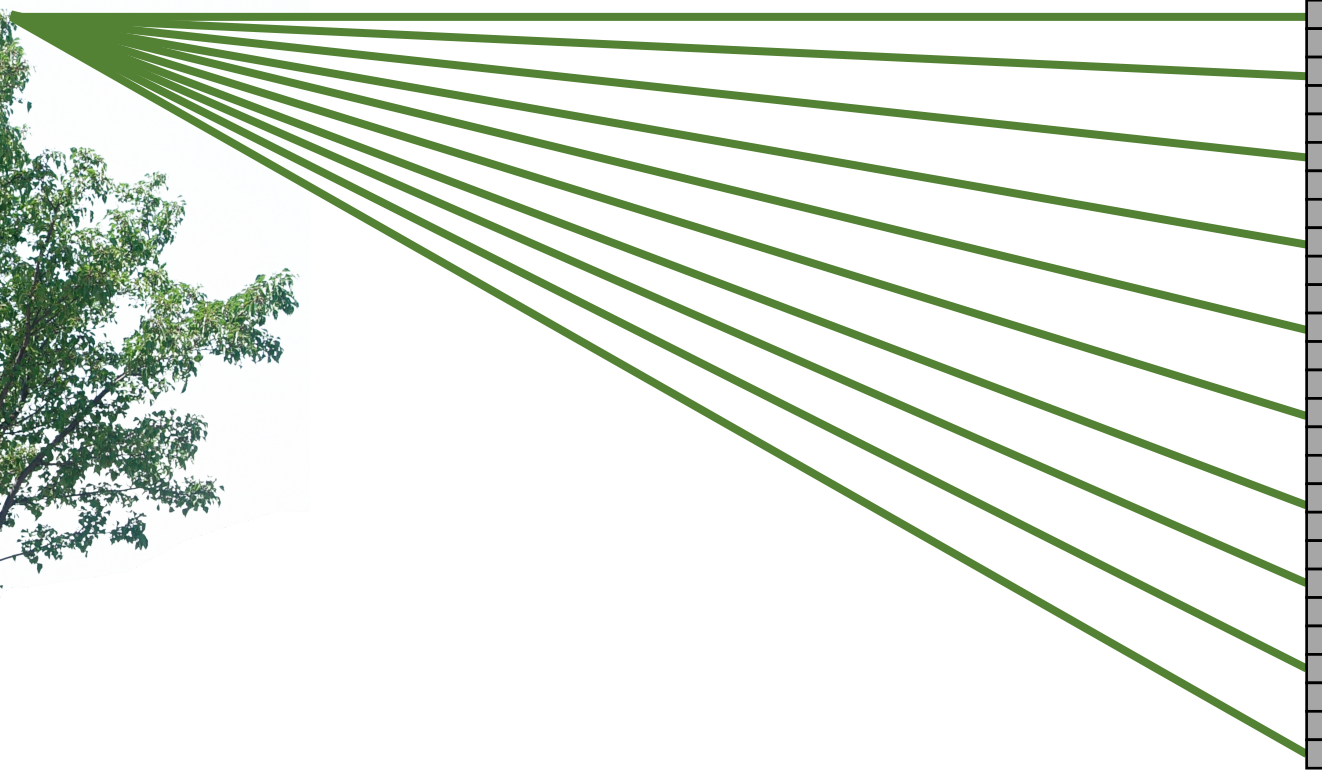


Bare-sensor imaging

real-world
object

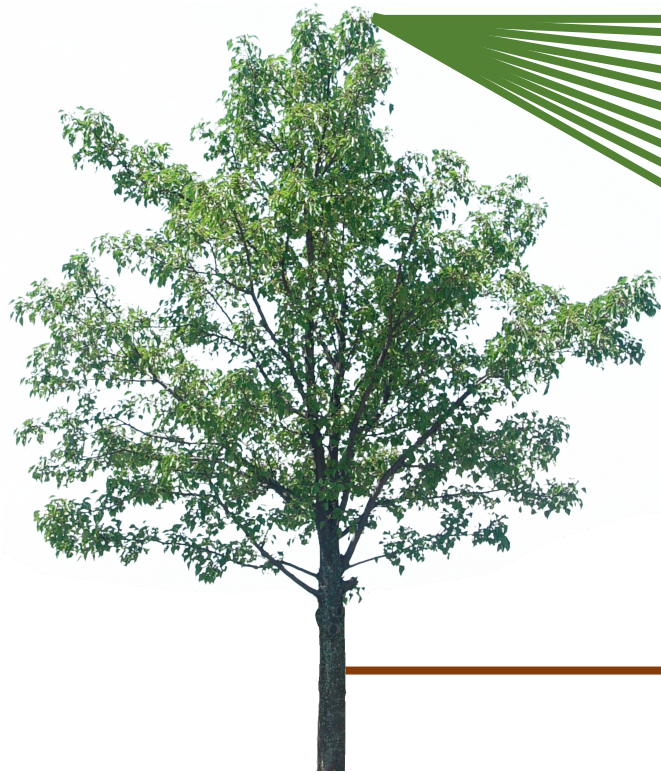


digital sensor
(CCD or CMOS)

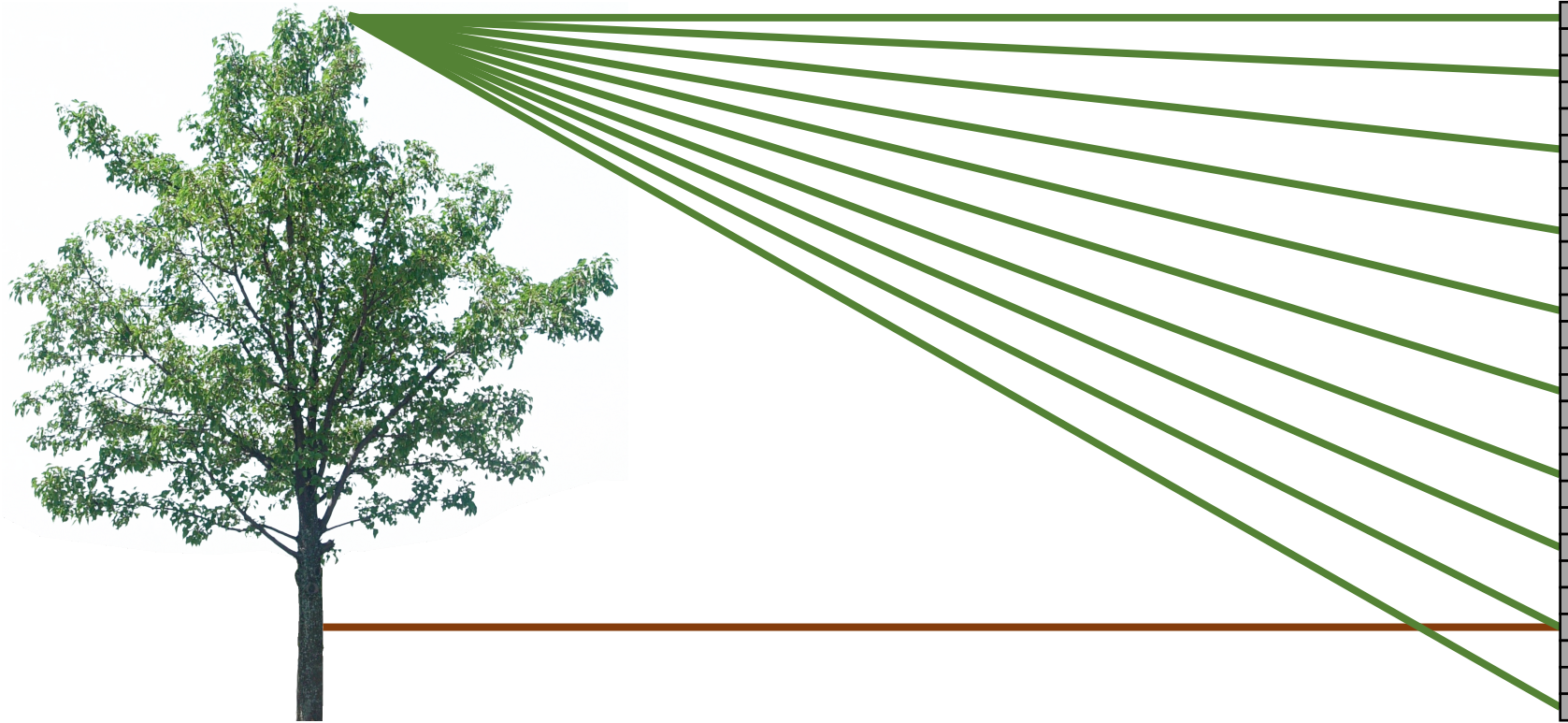


Bare-sensor imaging

real-world
object

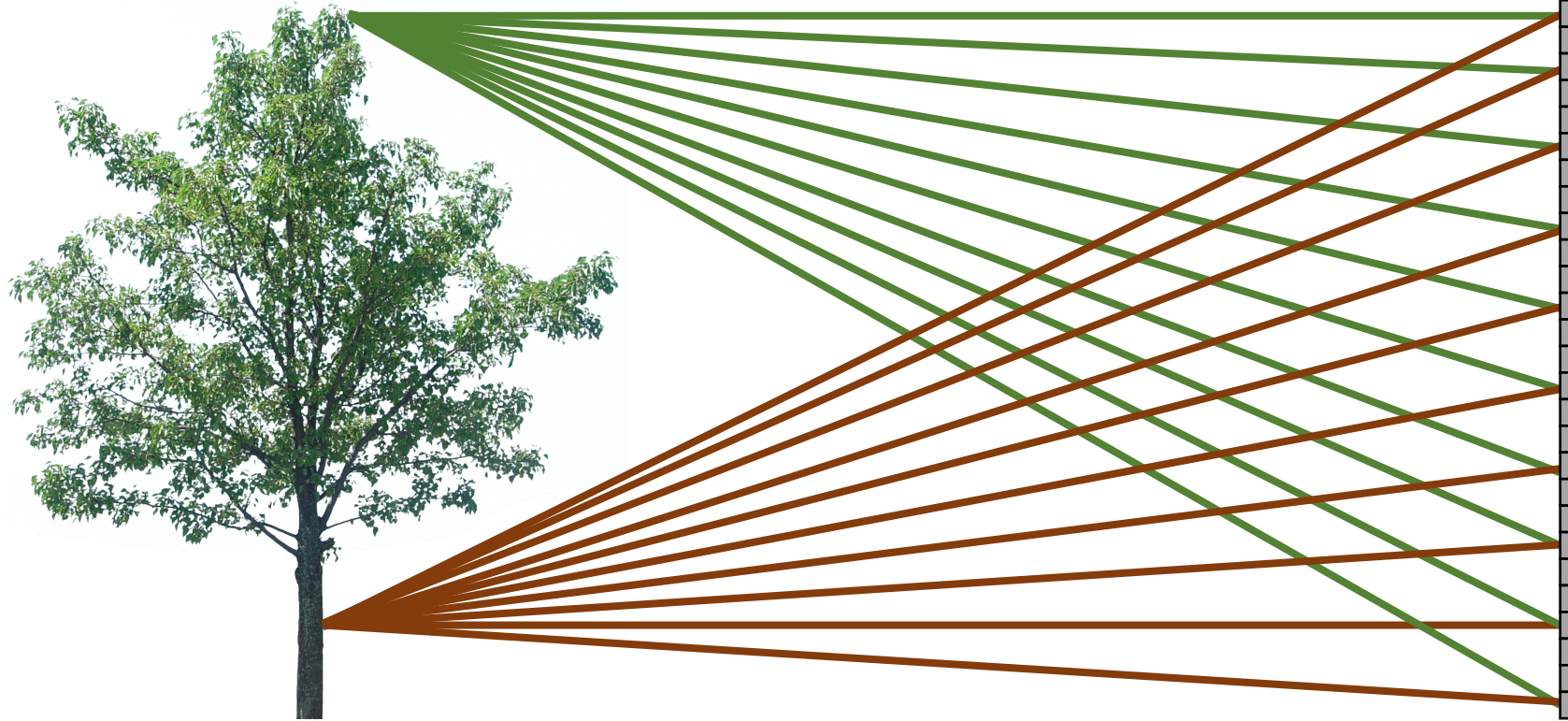


digital sensor
(CCD or CMOS)



Bare-sensor imaging

real-world
object



digital sensor
(CCD or CMOS)

All scene points contribute to all sensor pixels

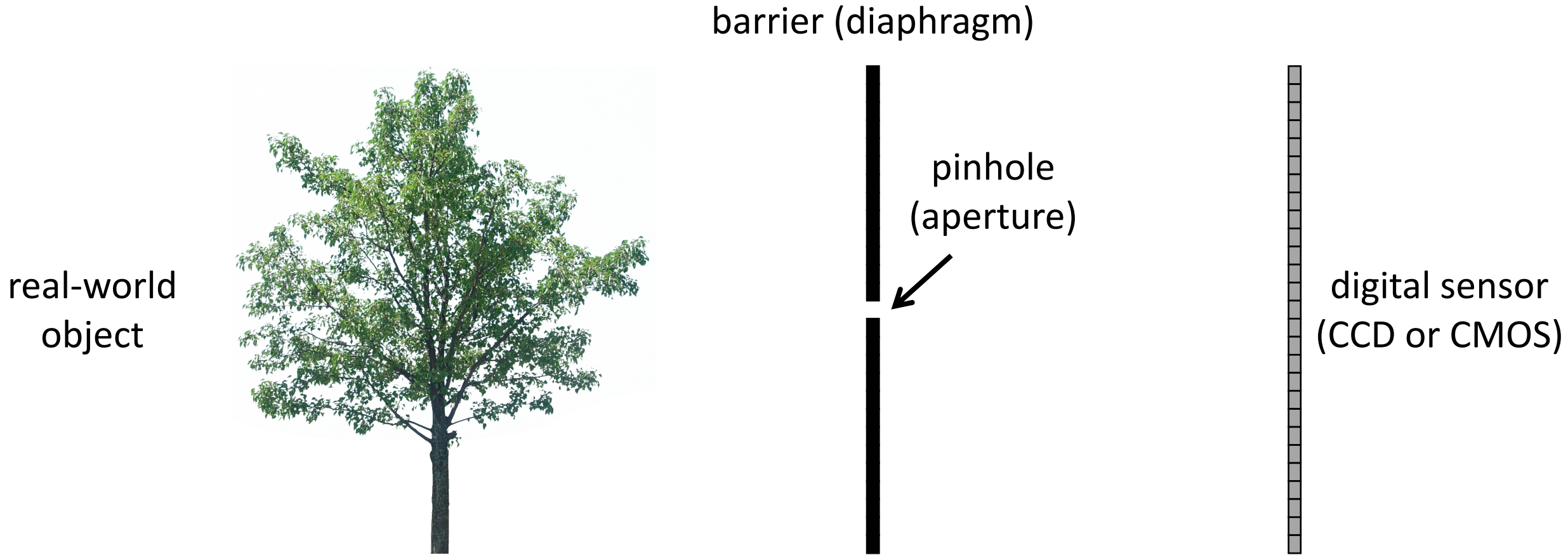
What does the
image on the
sensor look like?

Bare-sensor imaging



All scene points contribute to all sensor pixels

Let's add something to this scene



What would an image taken like this look like?

Pinhole imaging

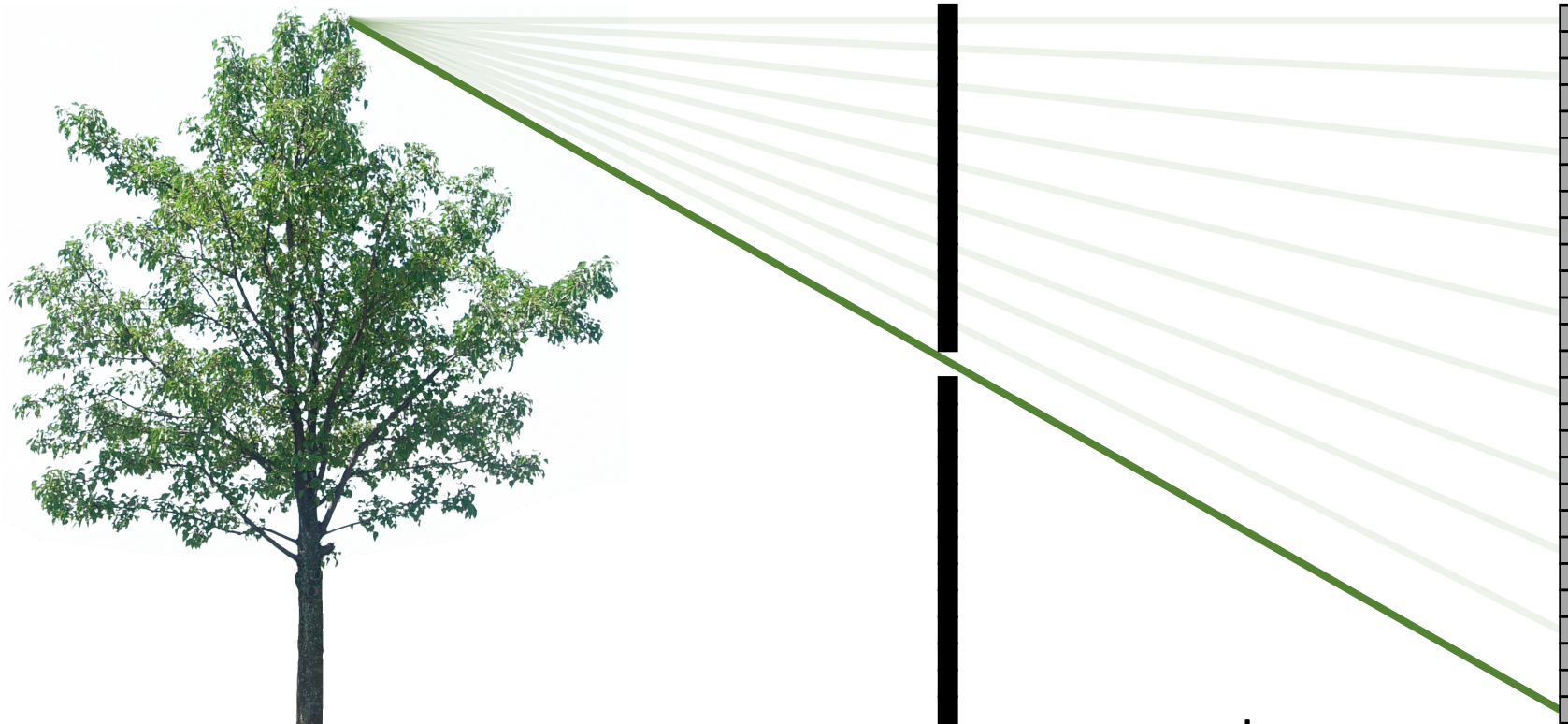
real-world
object



most rays
are blocked

one makes
it through

digital sensor
(CCD or CMOS)



Pinhole imaging

real-world
object

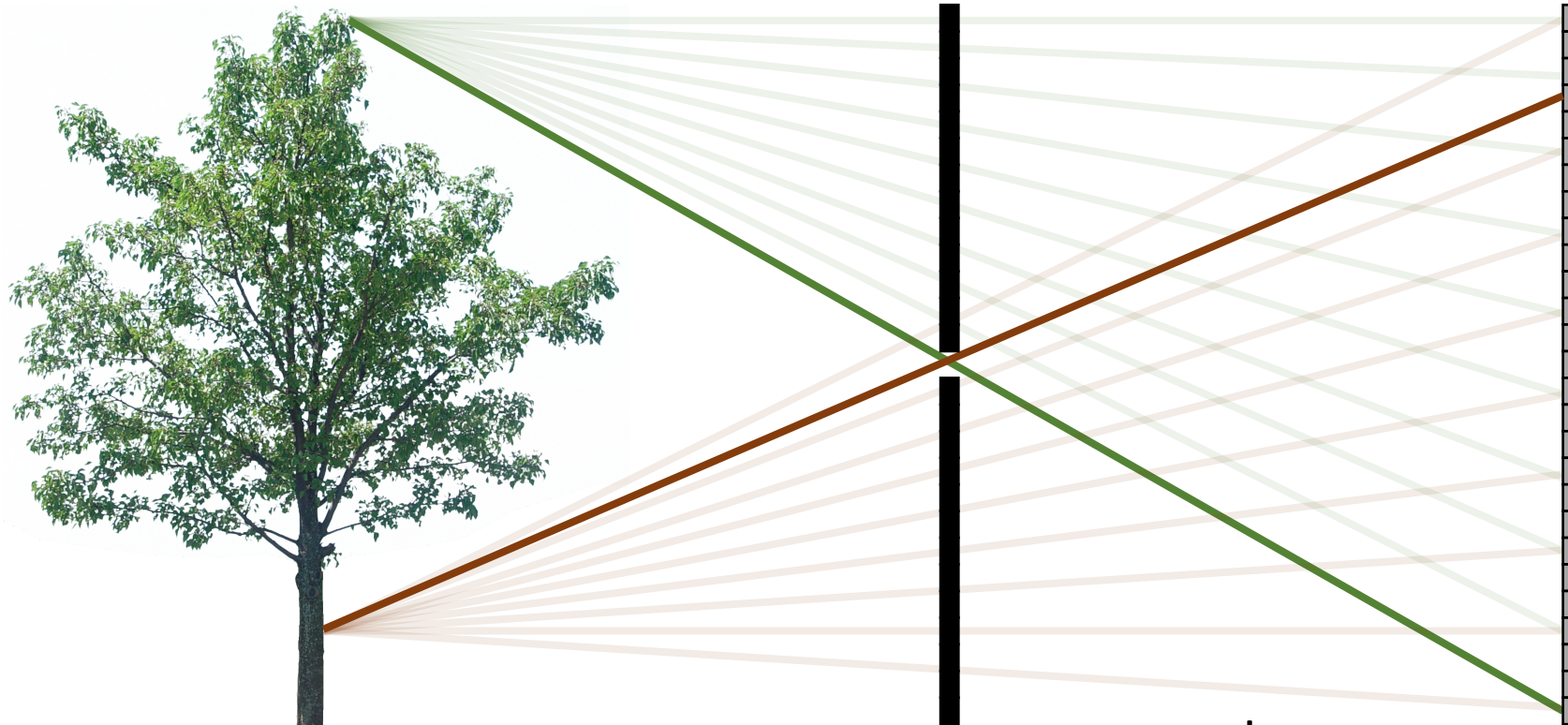


most rays
are blocked



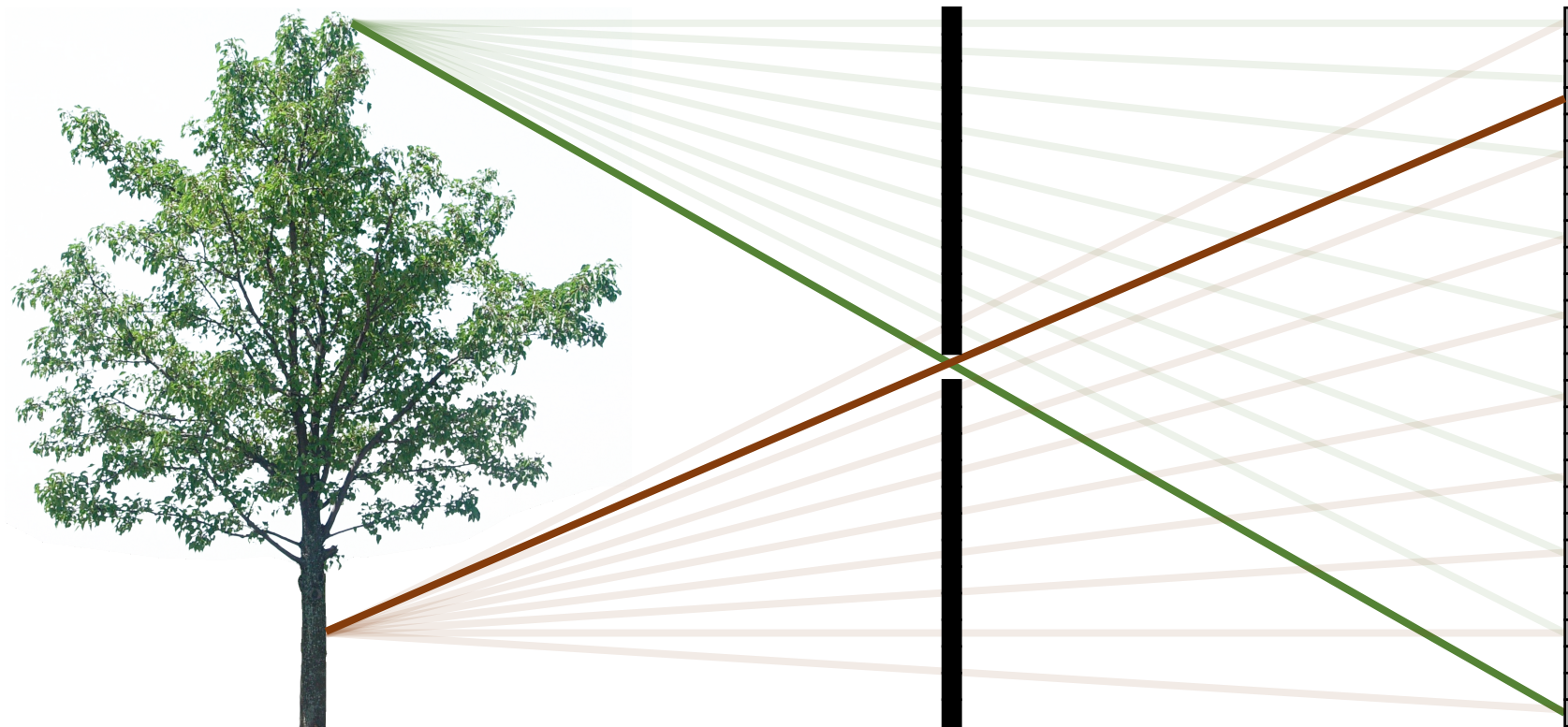
one makes
it through

digital sensor
(CCD or CMOS)



Pinhole imaging

real-world
object



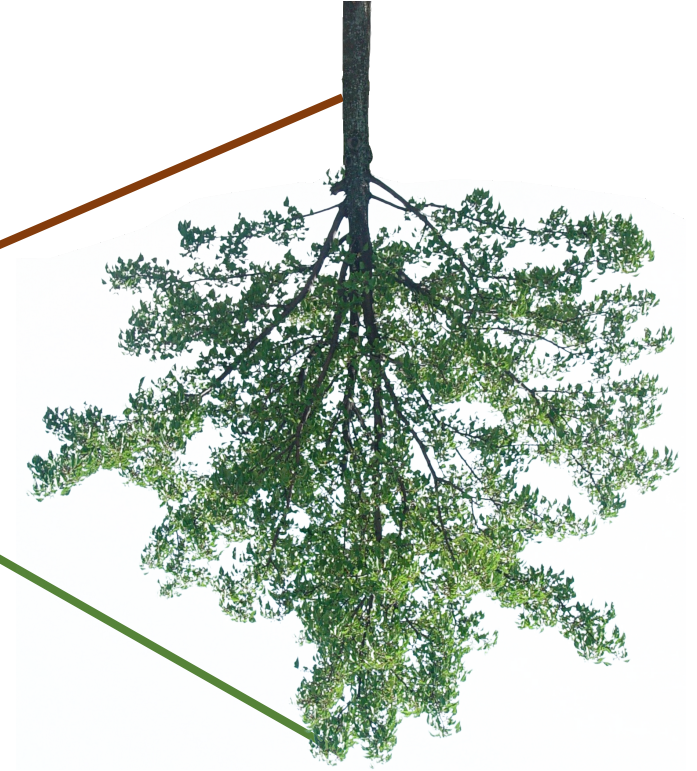
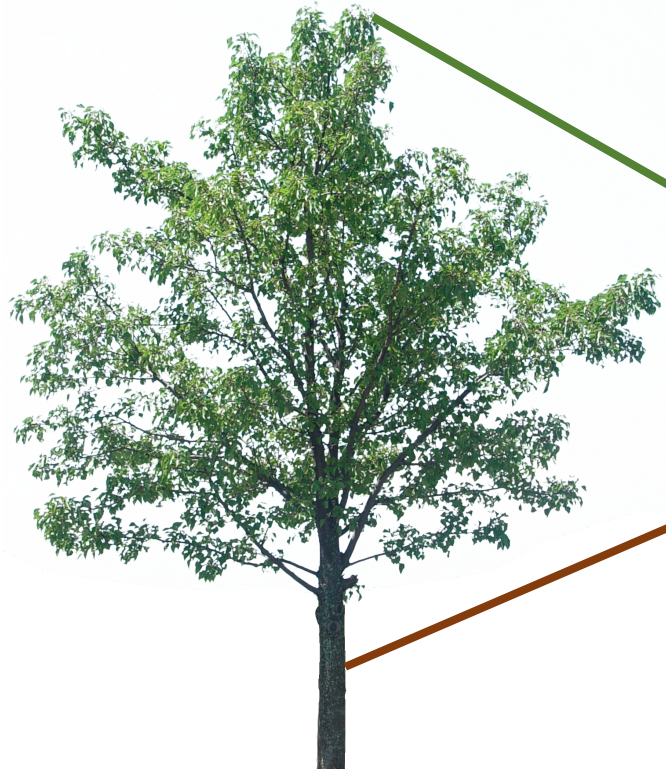
digital sensor
(CCD or CMOS)

Each scene point contributes to only one sensor pixel

What does the
image on the
sensor look like?

Pinhole imaging

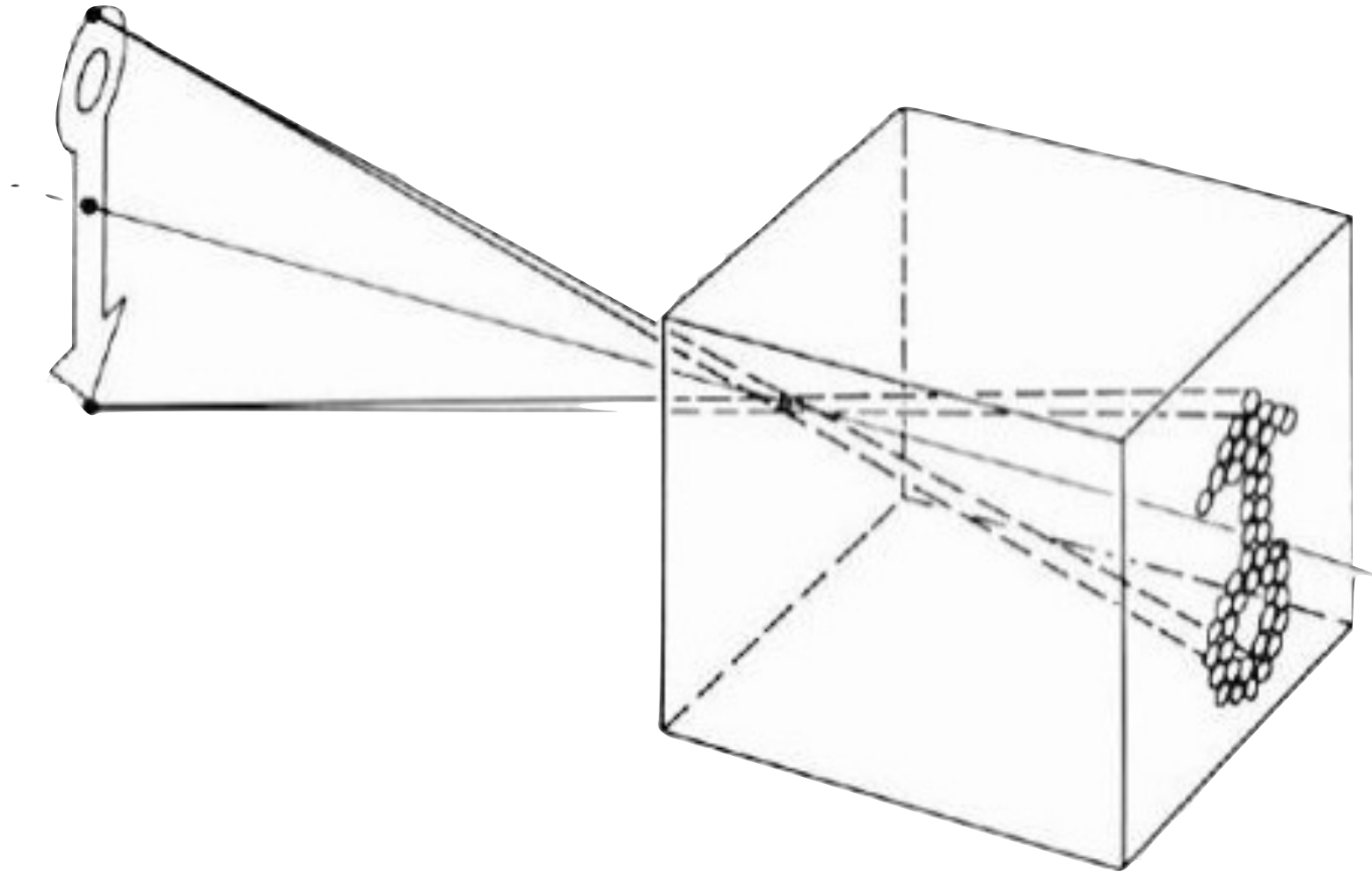
real-world
object



copy of real-world object
(inverted and scaled)

Pinhole camera

Pinhole camera a.k.a. camera obscura



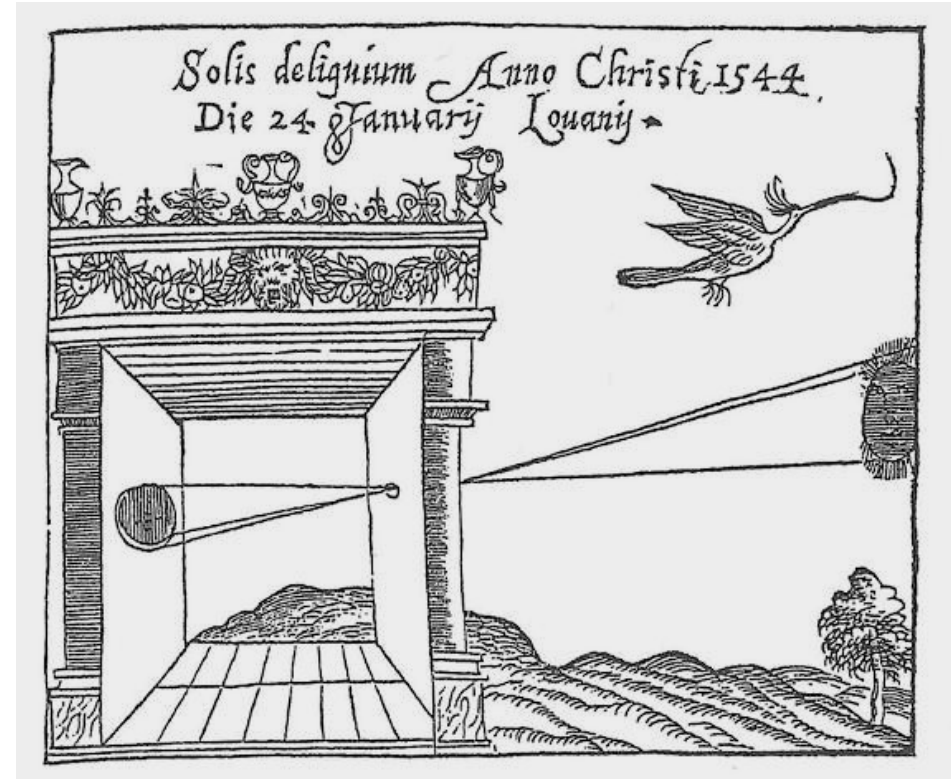
Pinhole camera a.k.a. camera obscura

First mention ...



Chinese philosopher Mozi
(470 to 390 BC)

First camera ...



Greek philosopher Aristotle
(384 to 322 BC)

Pinhole camera terms

real-world
object



barrier (diaphragm)



pinhole
(aperture)



digital sensor
(CCD or CMOS)

Pinhole camera terms

real-world
object



barrier (diaphragm)



pinhole
(aperture)



camera center
(center of projection)

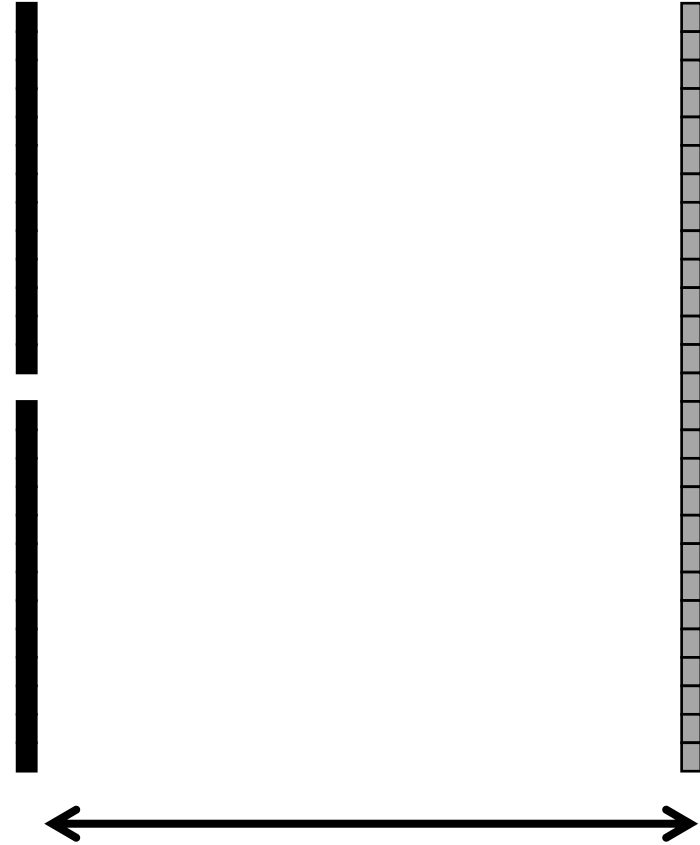
image plane



digital sensor
(CCD or CMOS)

Focal length

real-world
object

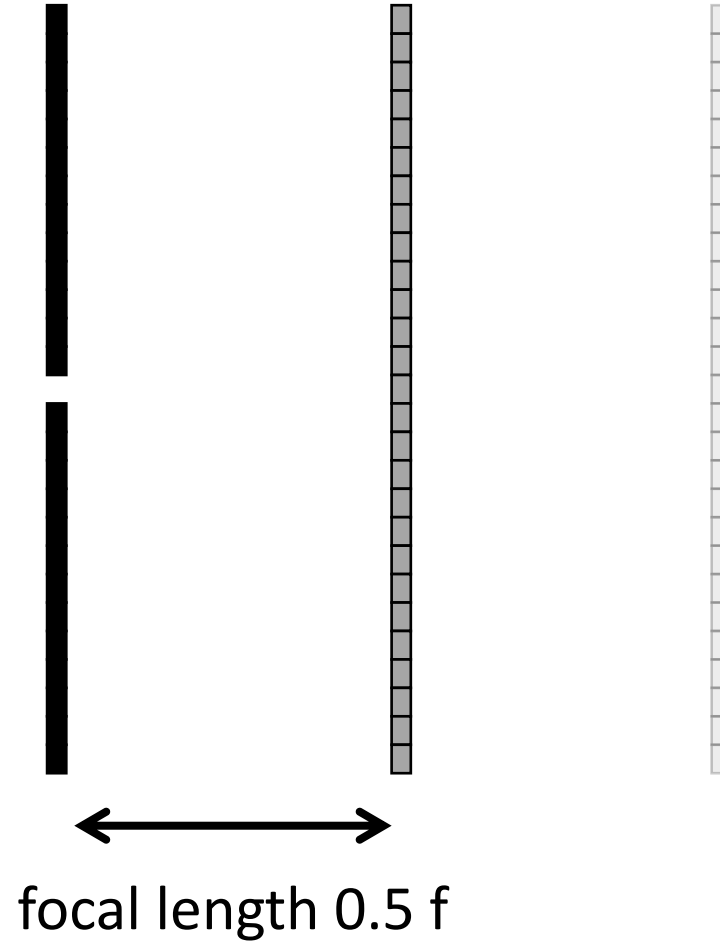


focal length f

Focal length

What happens as we change the focal length?

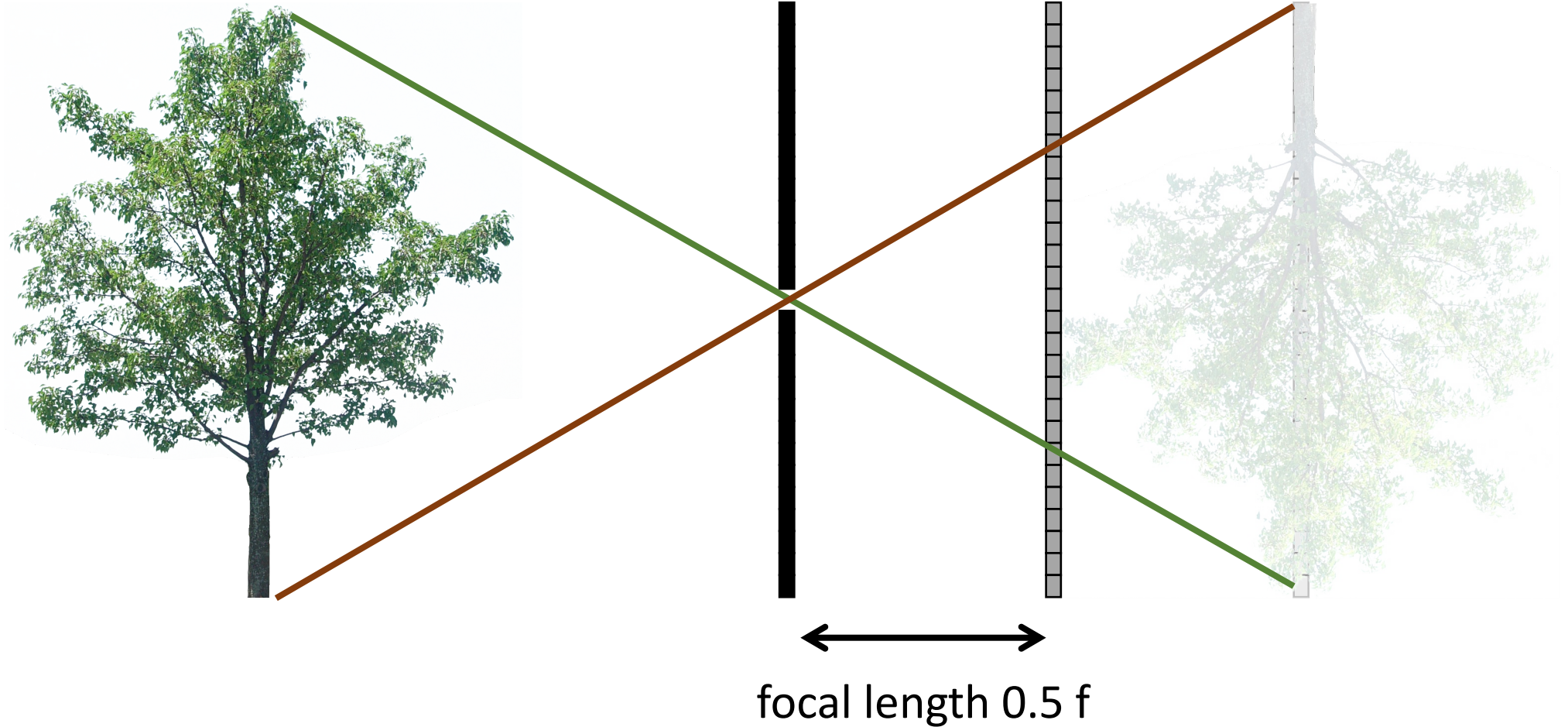
real-world
object



Focal length

What happens as we change the focal length?

real-world
object

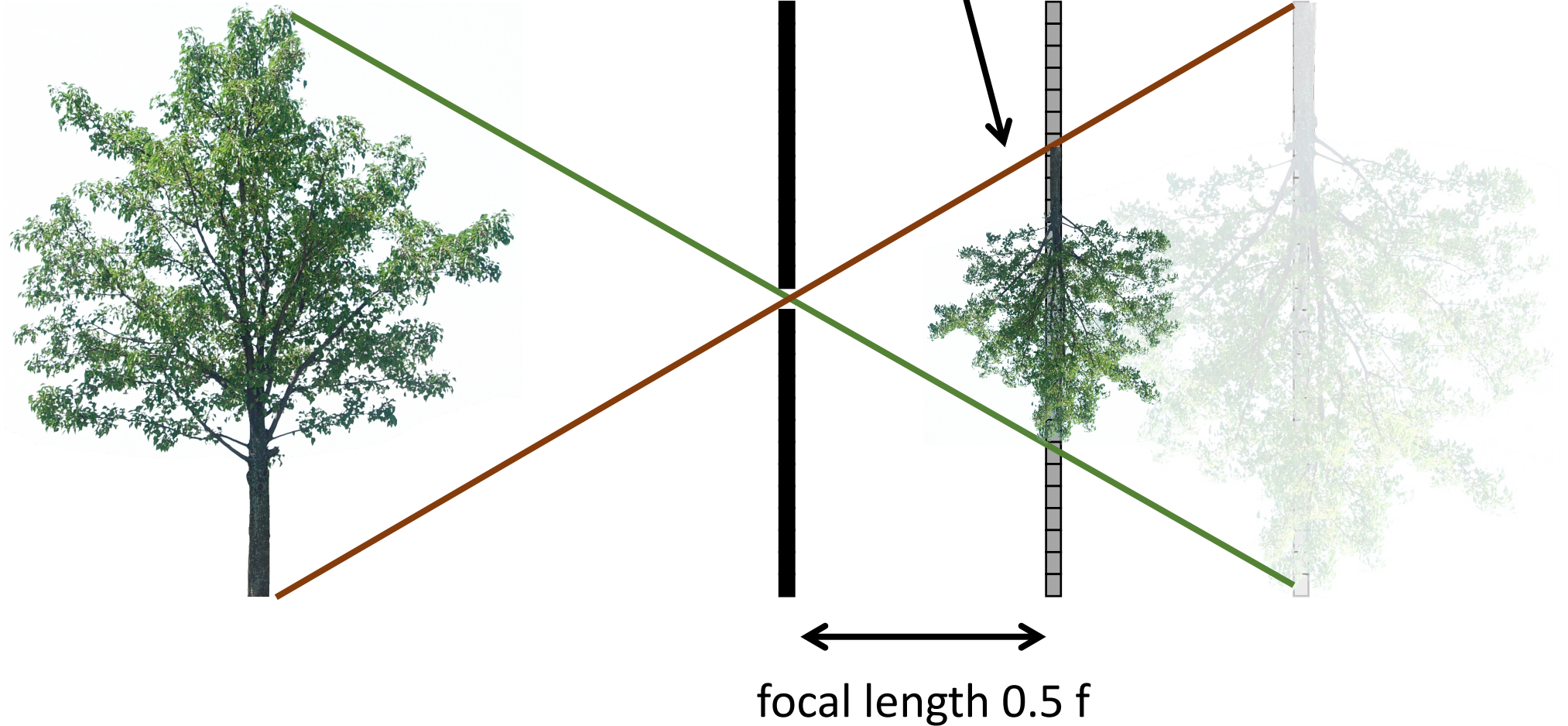


Focal length

What happens as we change the focal length?

object projection is half the size

real-world
object



Pinhole size

real-world
object



pinhole
diameter



Ideal pinhole has infinitesimally small size

- In practice that is impossible.

Pinhole size

What happens as we change the pinhole diameter?

real-world
object



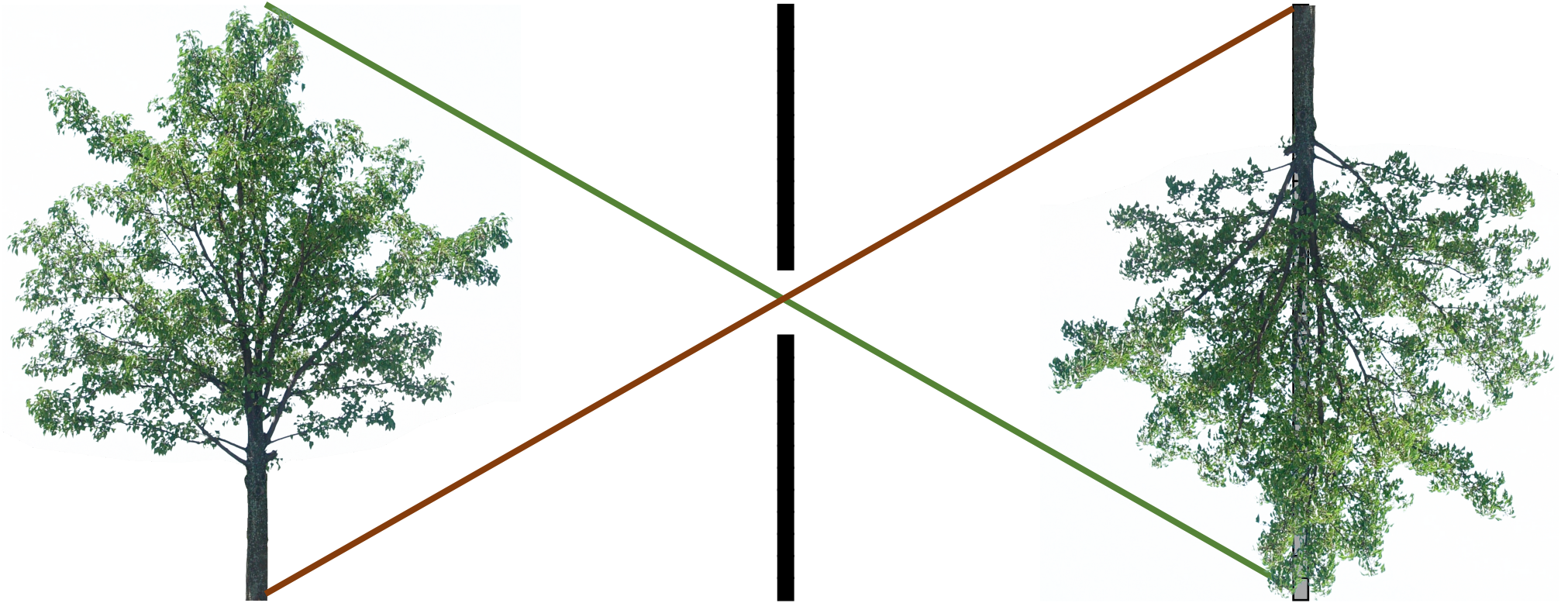
pinhole
diameter



Pinhole size

What happens as we change the pinhole diameter?

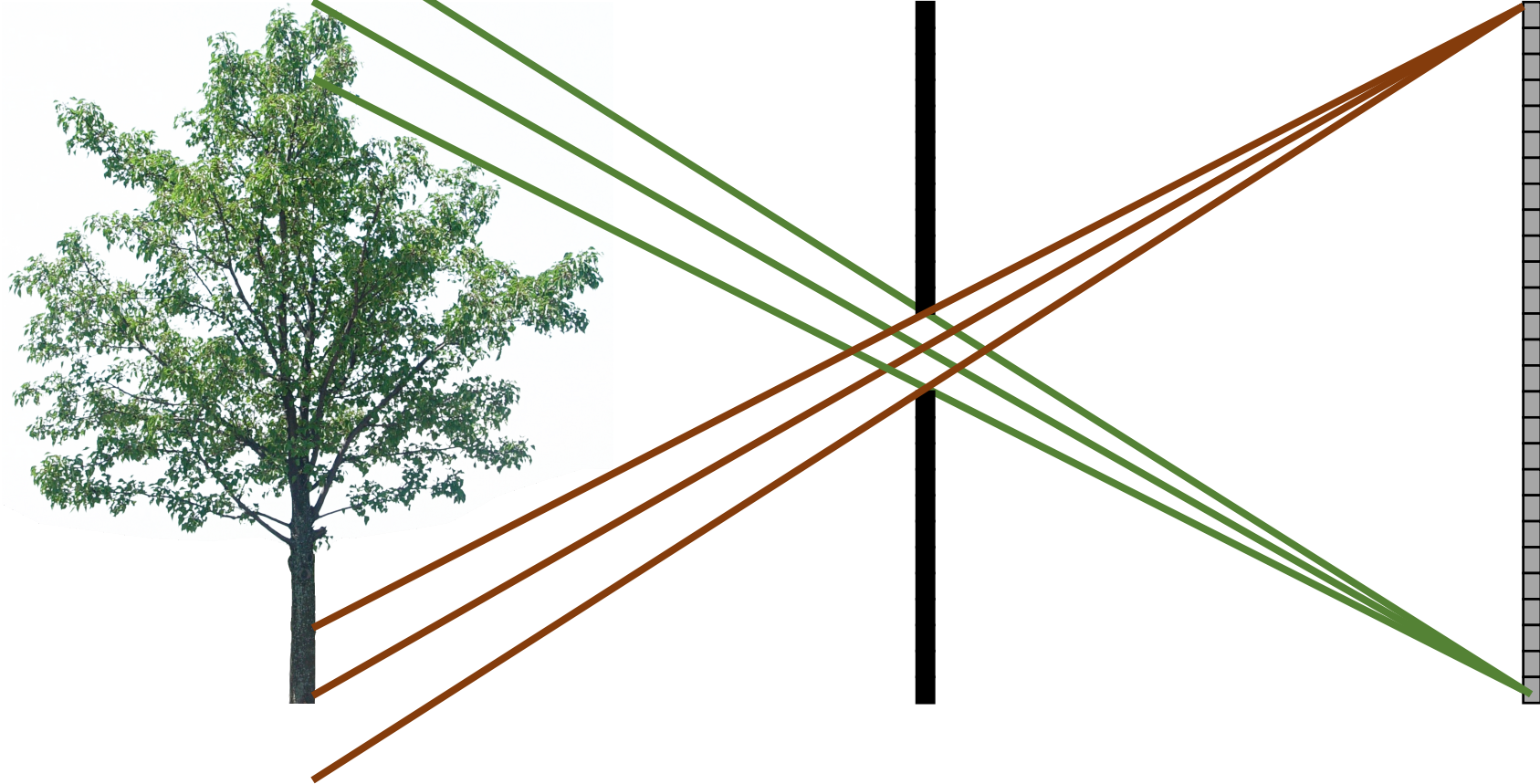
real-world
object



Pinhole size

What happens as we change the pinhole diameter?

real-world
object

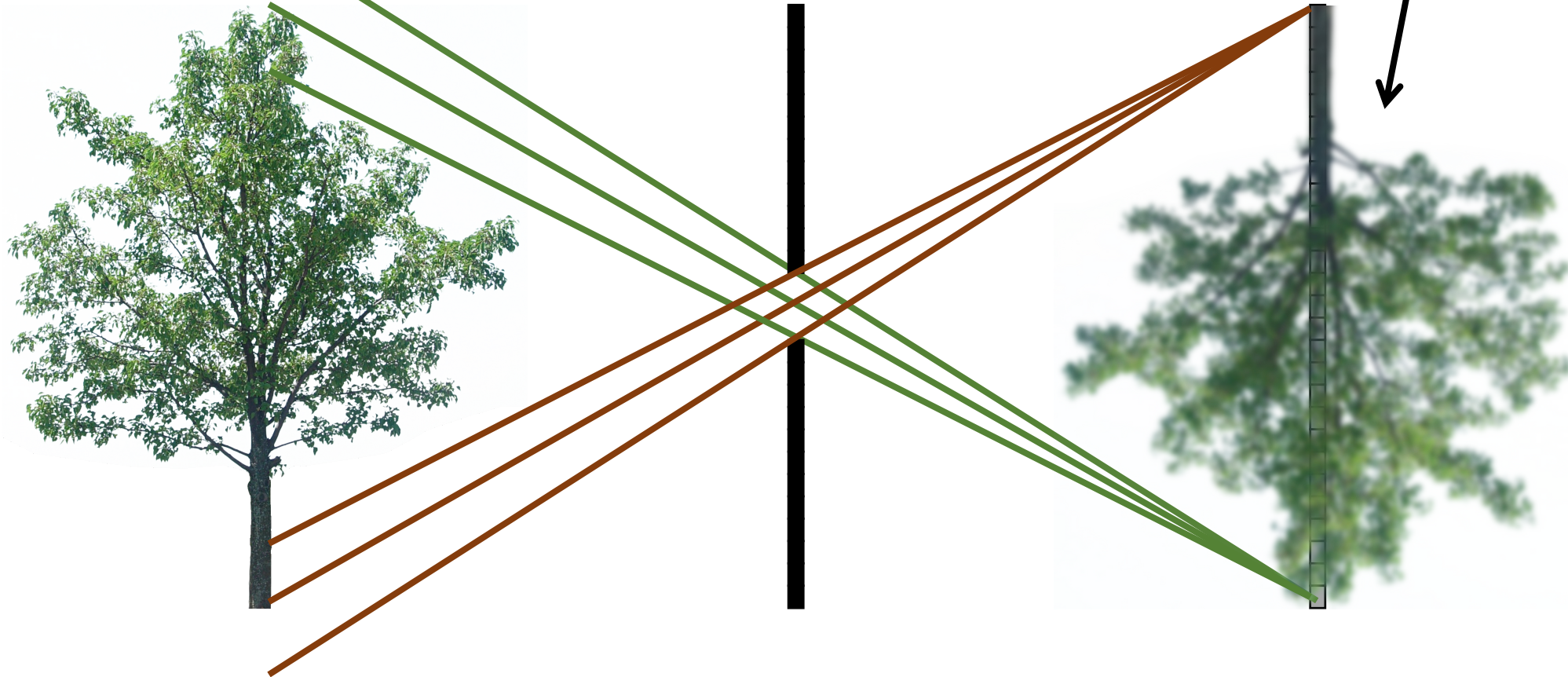


Pinhole size

What happens as we change the pinhole diameter?

object projection becomes blurrier

real-world object

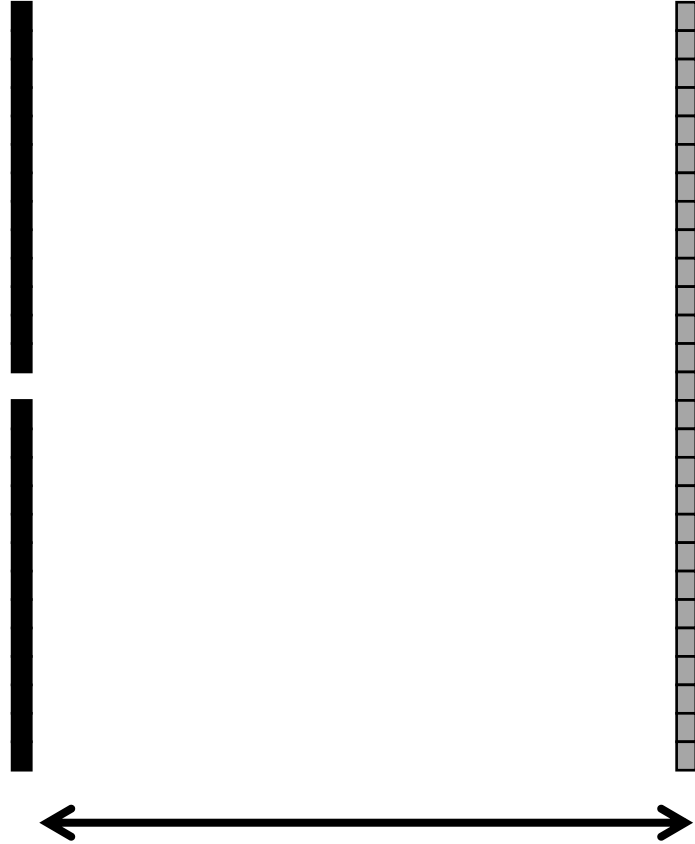


What about light efficiency?

real-world
object



pinhole
diameter



focal length f

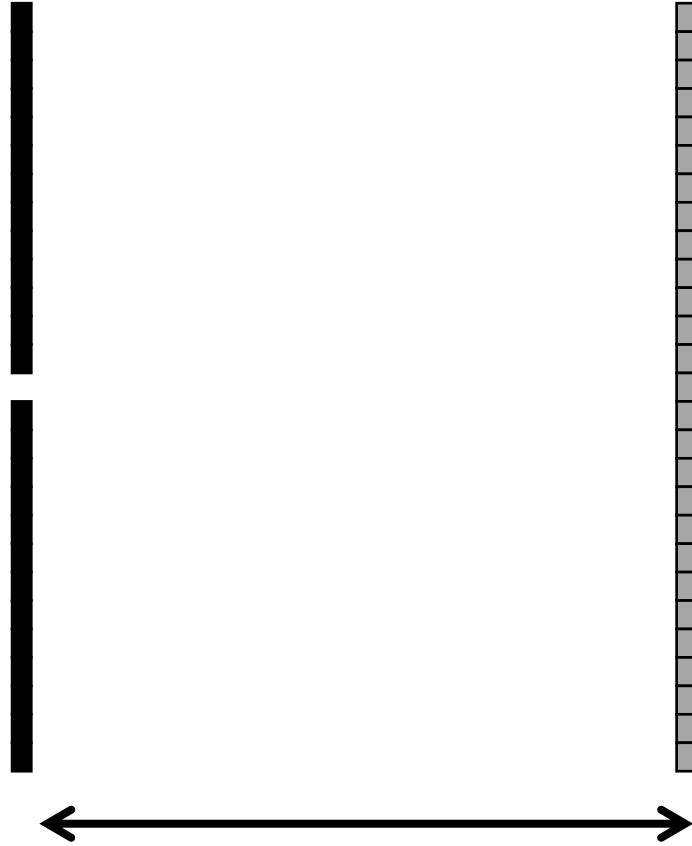
- What is the effect of doubling the pinhole diameter?
- What is the effect of doubling the focal length?

What about light efficiency?

real-world
object



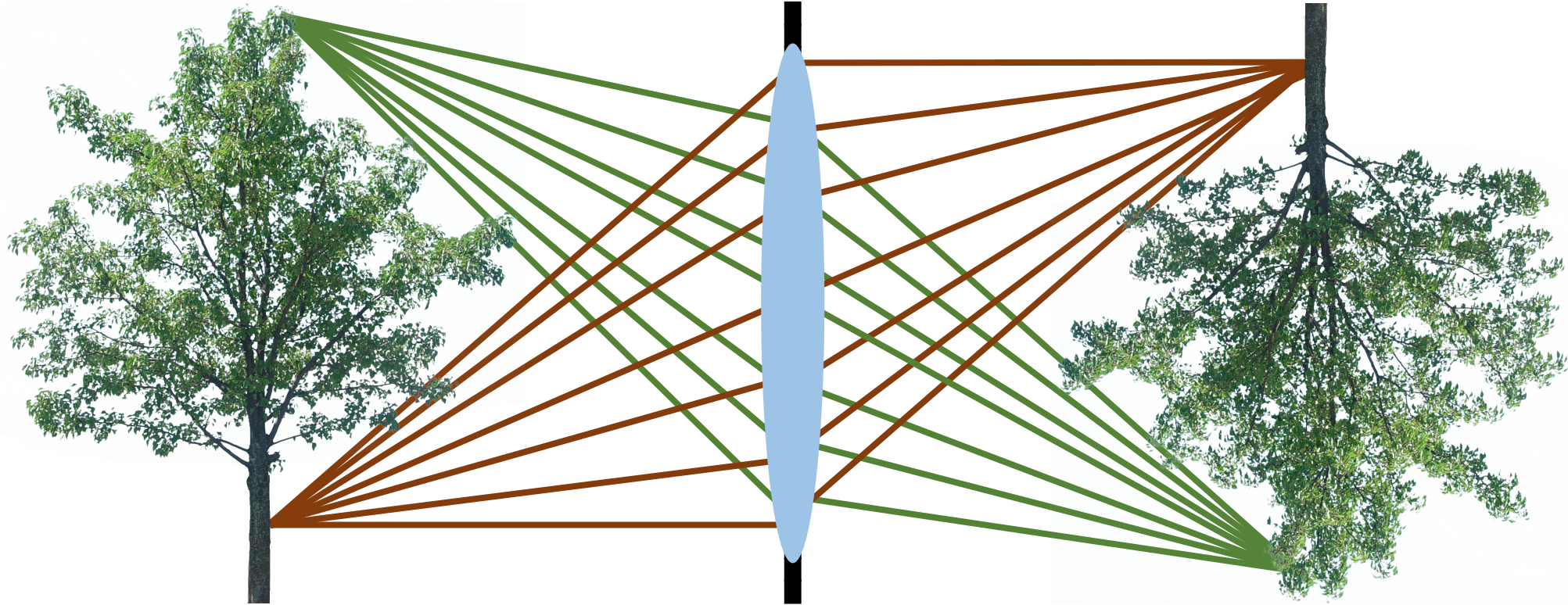
pinhole
diameter



focal length f

- 2x pinhole diameter \rightarrow 4x light
- 2x focal length \rightarrow $\frac{1}{4}$ x light

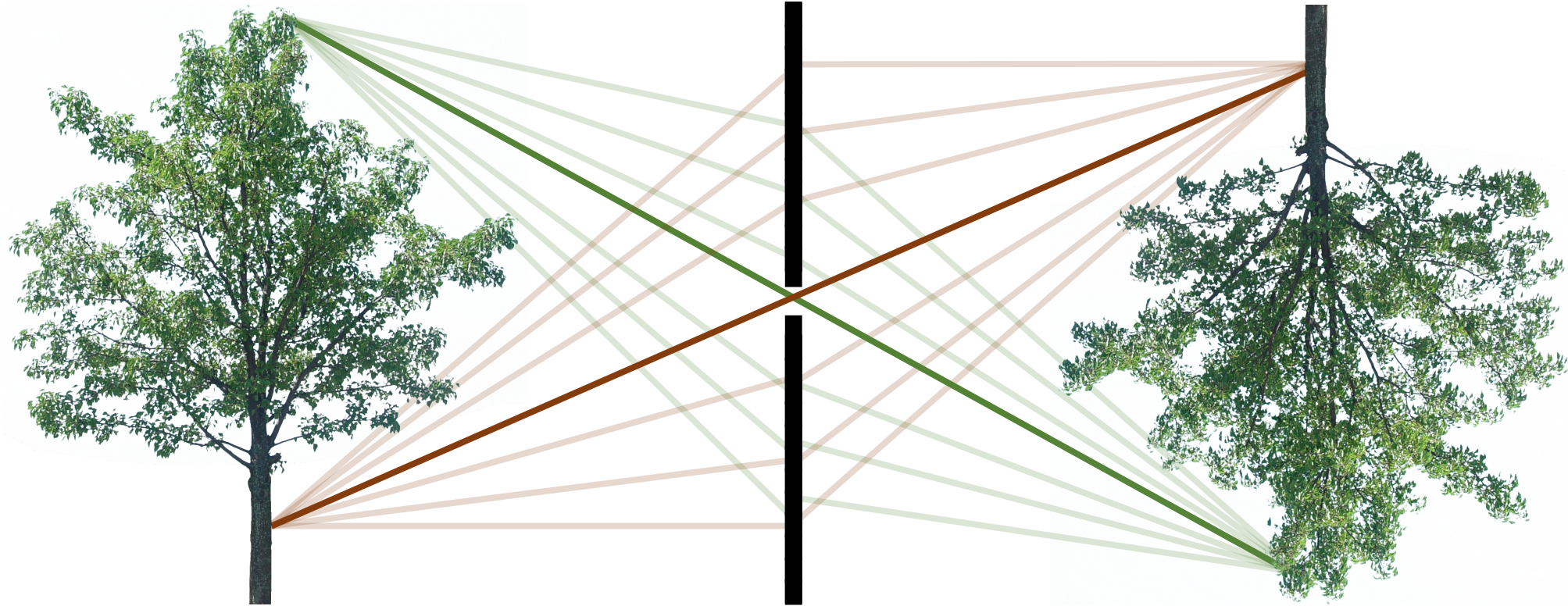
The lens camera



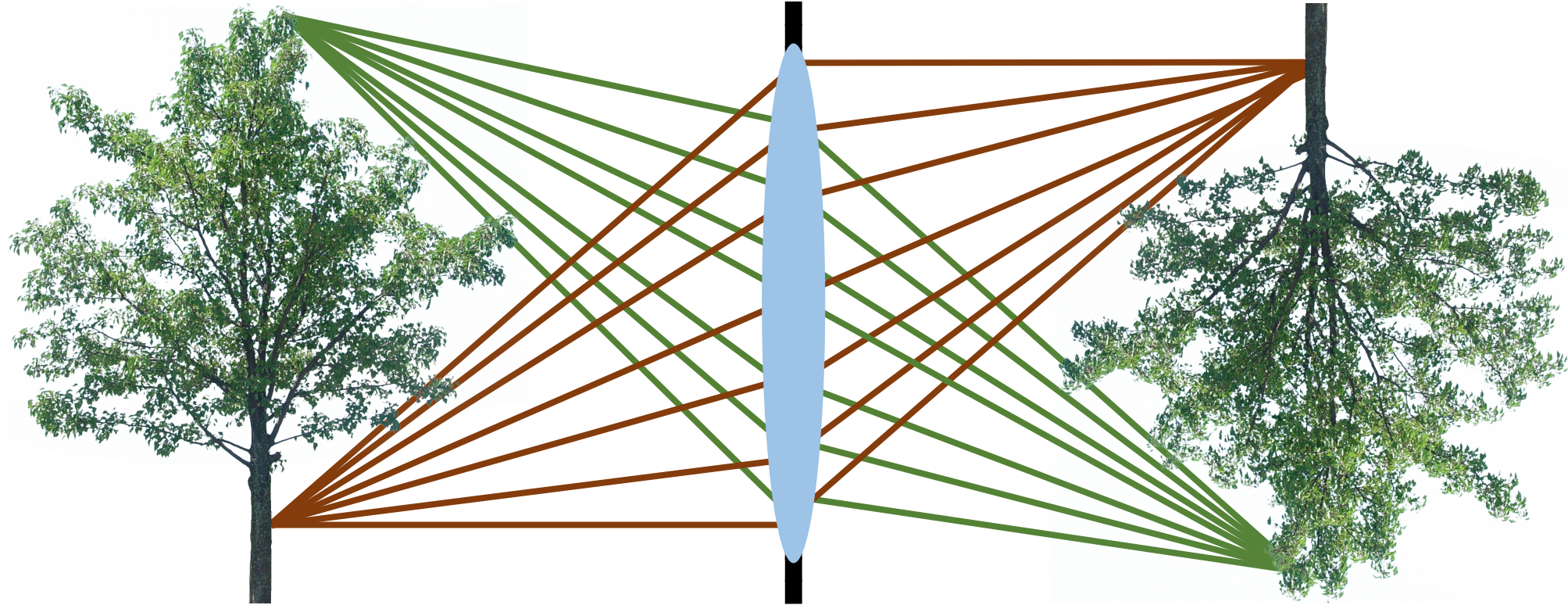
Lenses map “bundles” of rays from points on the scene to the sensor.

How does this mapping work exactly?

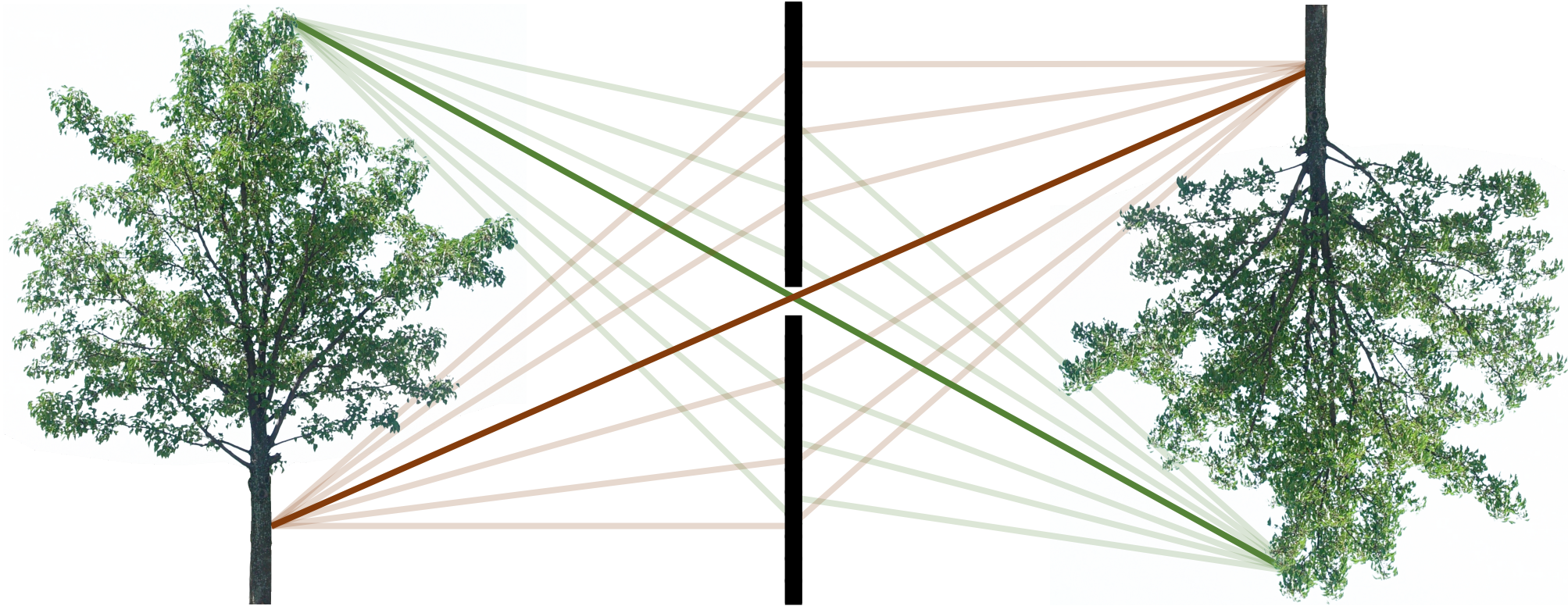
The pinhole camera



The lens camera

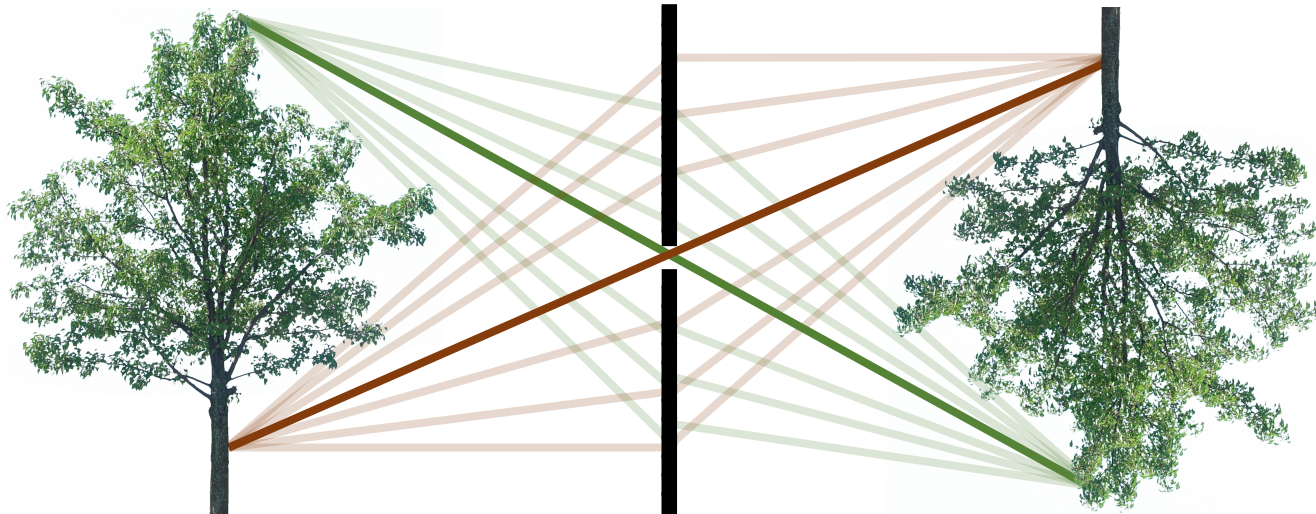
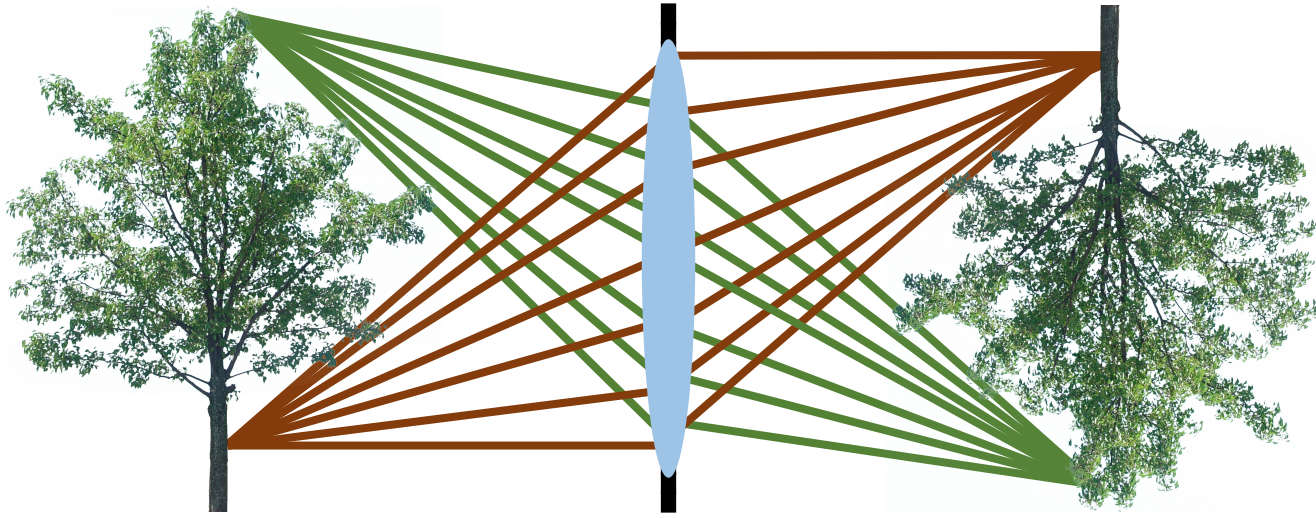


The pinhole camera



Central rays propagate in the same way for both models!

Describing both lens and pinhole cameras

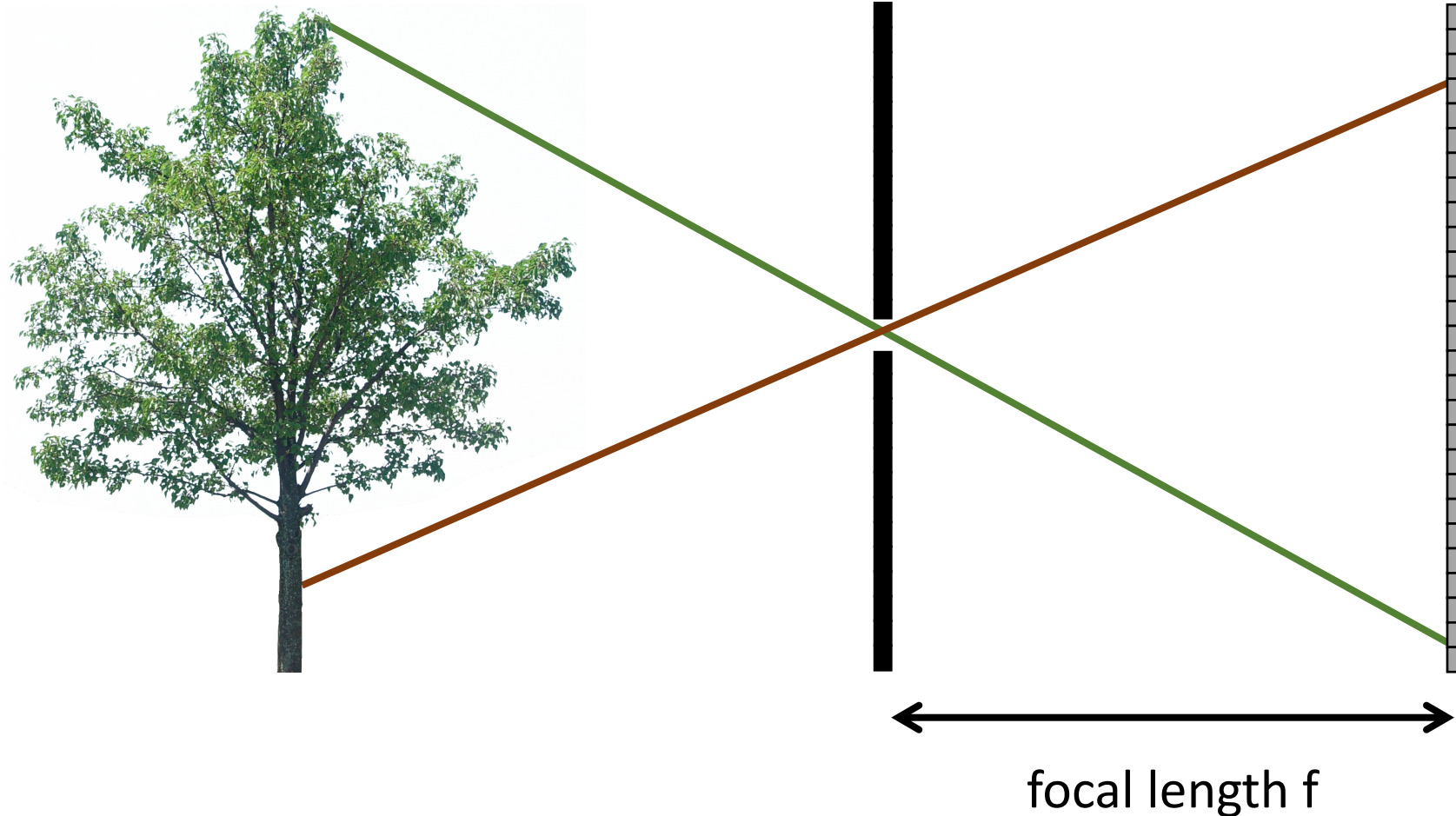


We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.

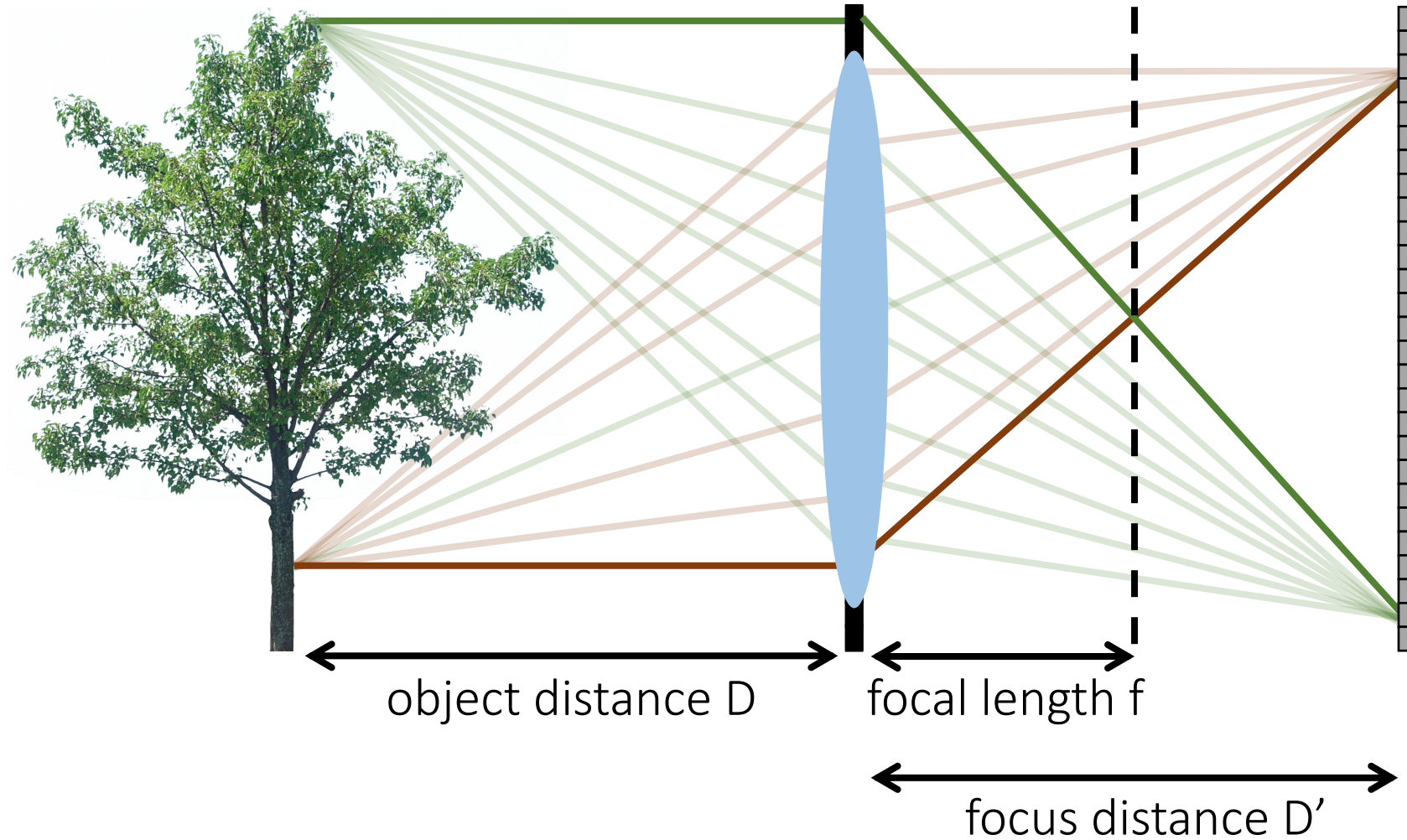
Important difference: focal length

In a pinhole camera, focal length is distance between aperture and sensor

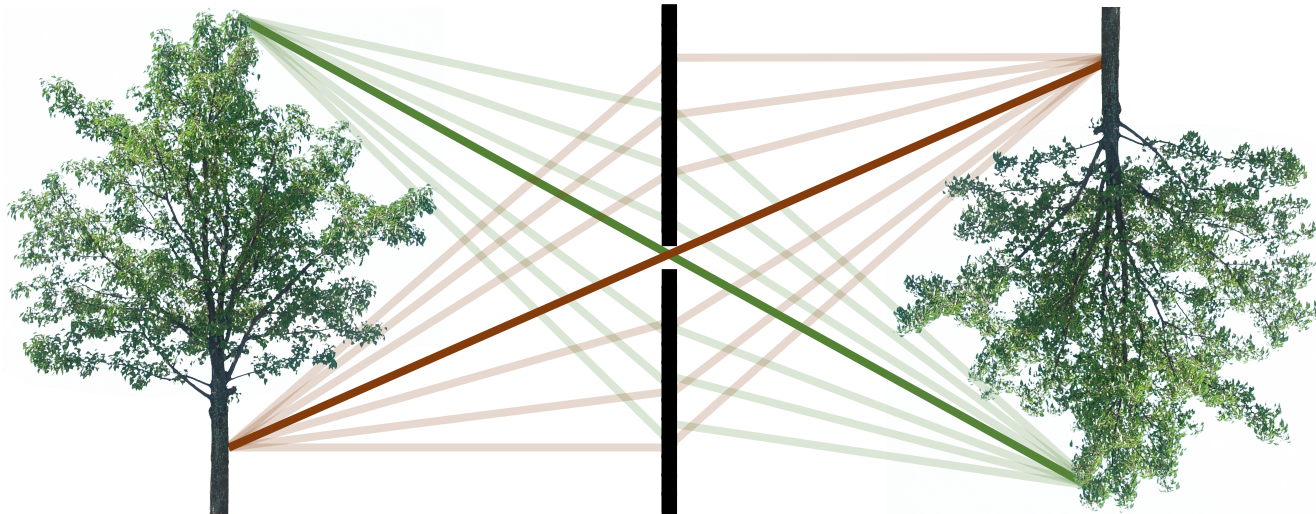
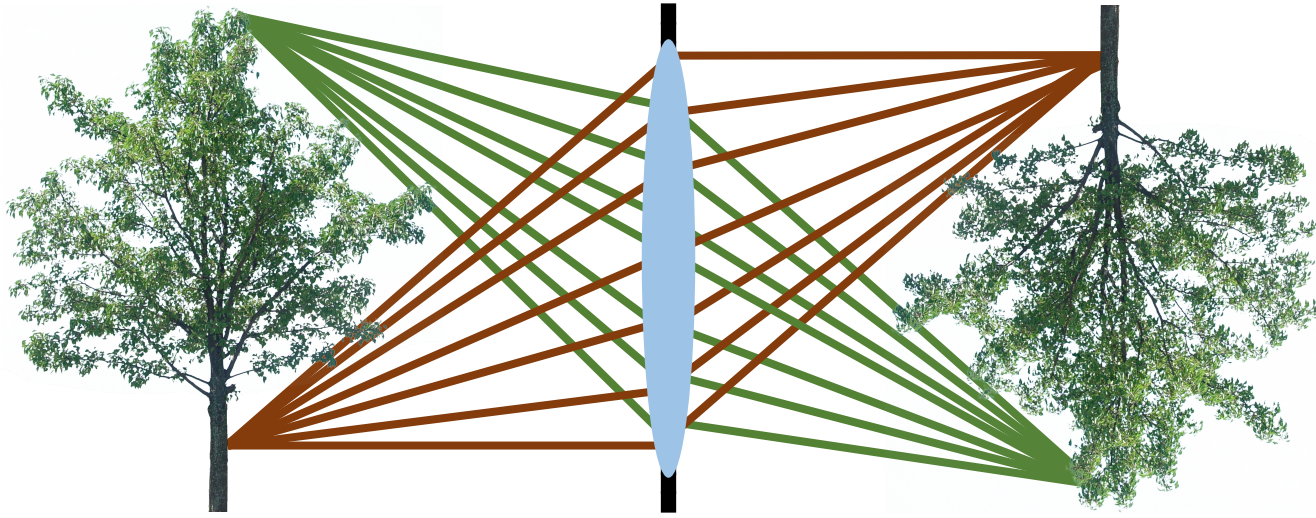


Important difference: focal length

In a lens camera, focal length is distance where parallel rays intersect



Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.
- We assume that the focus distance of the lens camera is equal to the focal length of the pinhole camera.

Remember: *focal length* f refers to different things for lens and pinhole cameras.

- In this lecture, we use it to refer to the aperture-sensor distance, as in the pinhole camera case.

Accidental pinholes





What does this image say about the world outside?



Accidental pinhole camera



Antonio Torralba, William T. Freeman
Computer Science and Artificial Intelligence Laboratory (CSAIL)
MIT
torralba@mit.edu, billf@mit.edu

Accidental pinhole camera

projected pattern on the wall



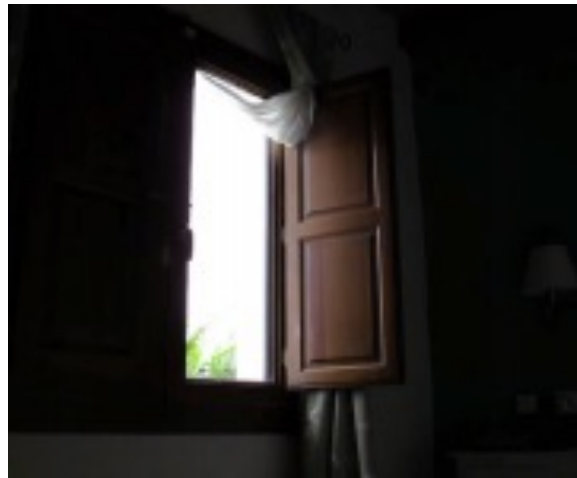
upside down



window with smaller gap



view outside window



window is an aperture

Pinhole cameras

What are we imaging here?



Camera matrix

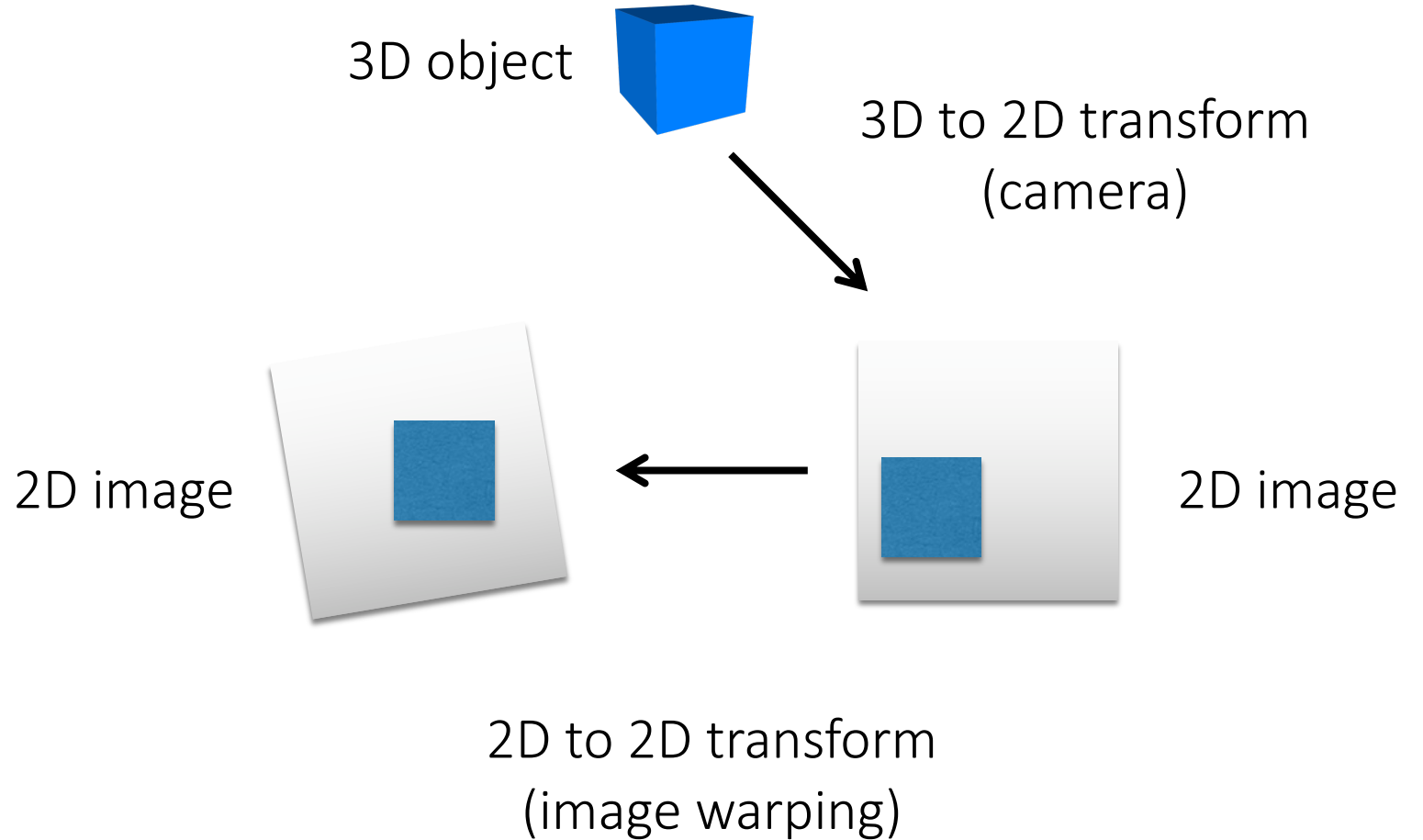
The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image



The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

homogeneous coordinates

$$\mathbf{x} = \mathbf{P} \mathbf{X}$$

2D image point camera matrix 3D world point

What are the dimensions of each variable?

The camera as a coordinate transformation

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

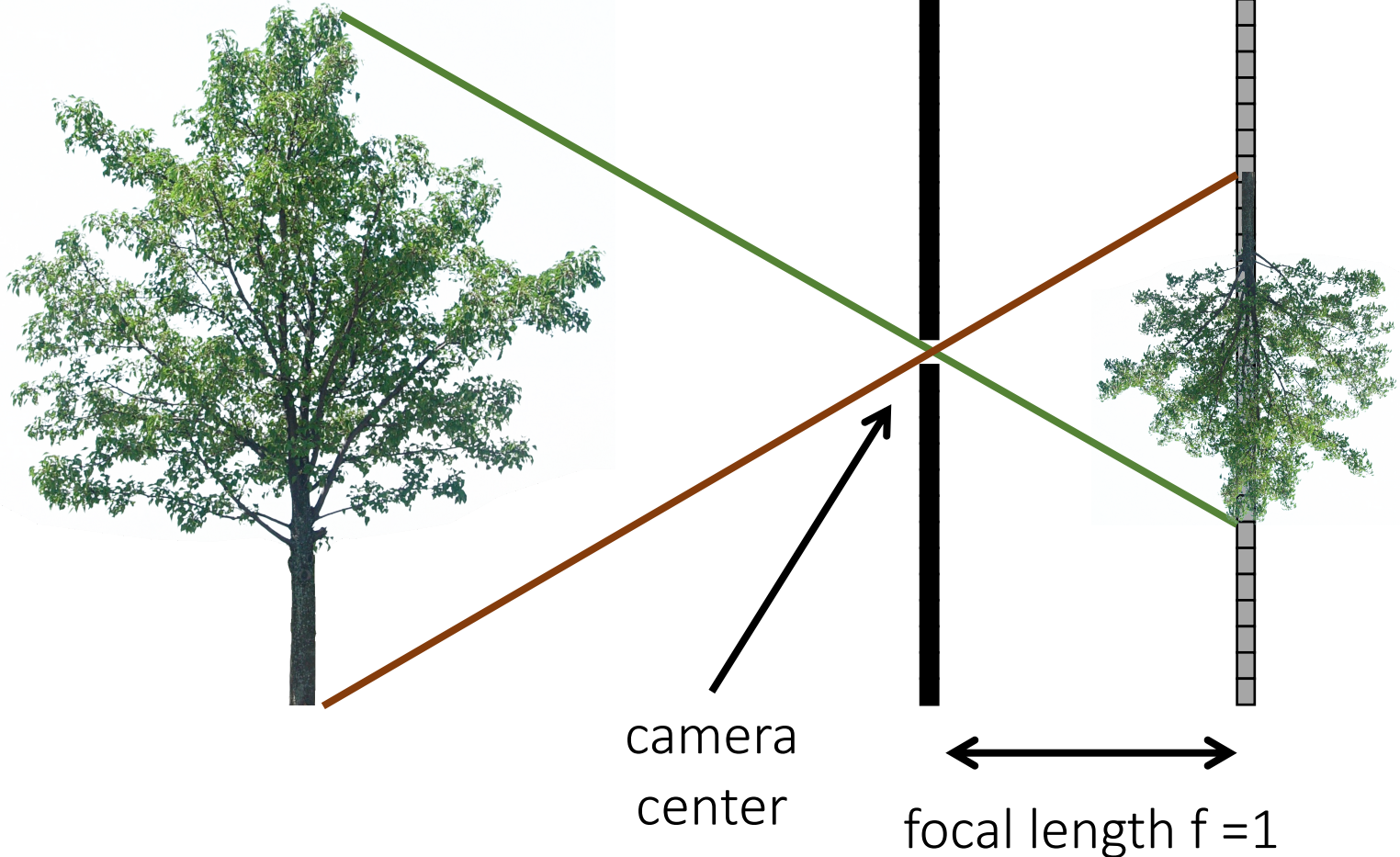
homogeneous
image coordinates
3 x 1

camera
matrix
3 x 4

homogeneous
world coordinates
4 x 1

The pinhole camera

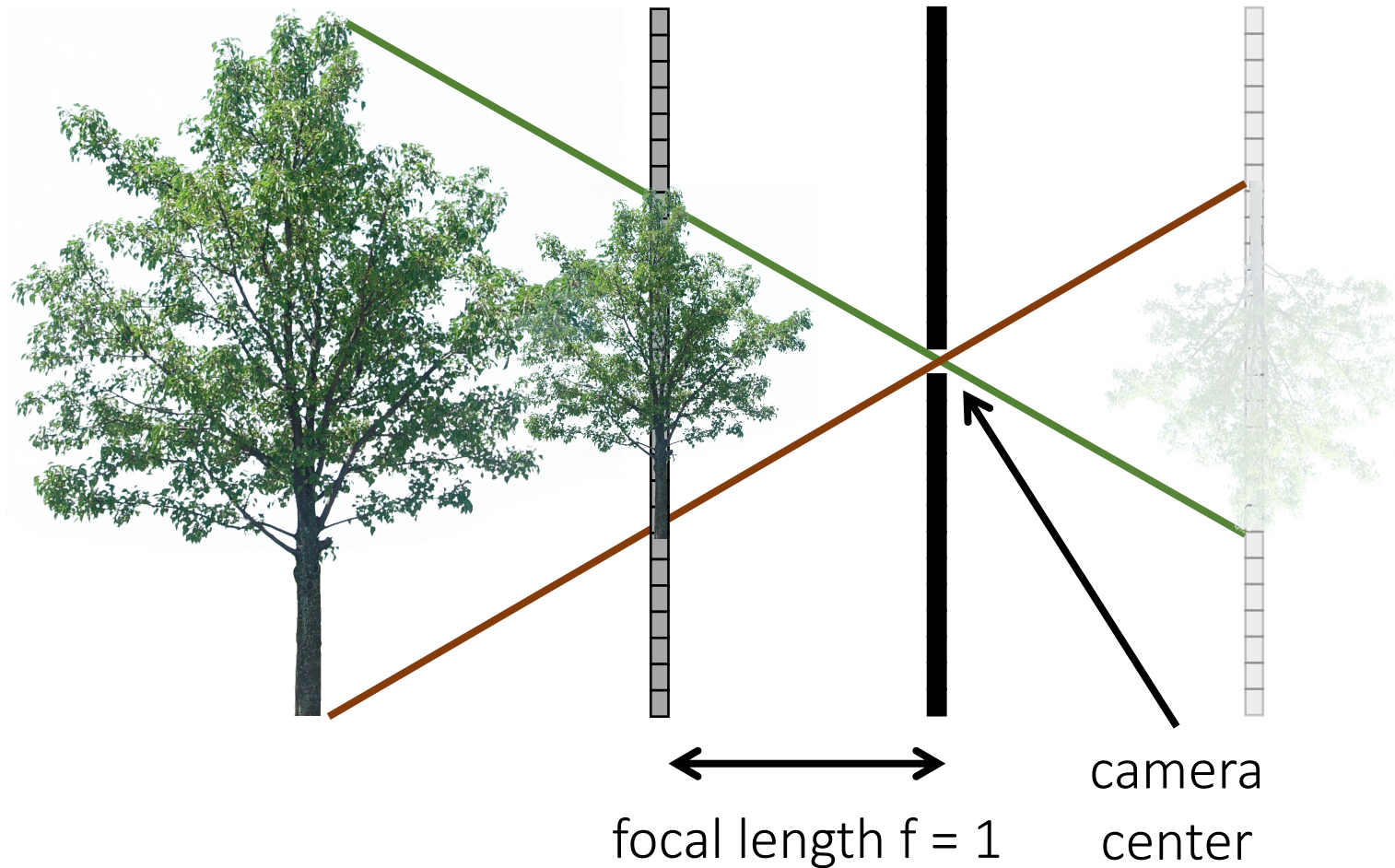
real-world
object



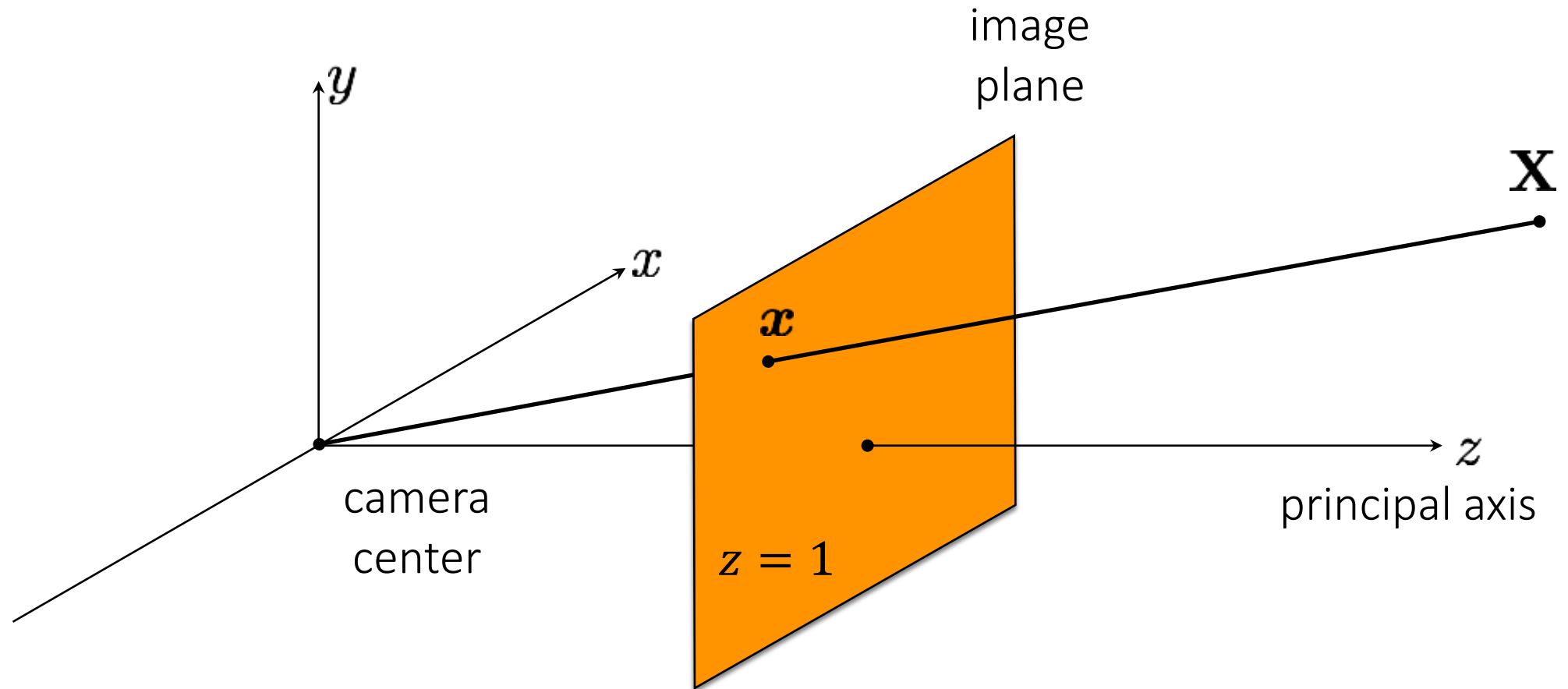
The (rearranged) pinhole camera

image plane

real-world
object

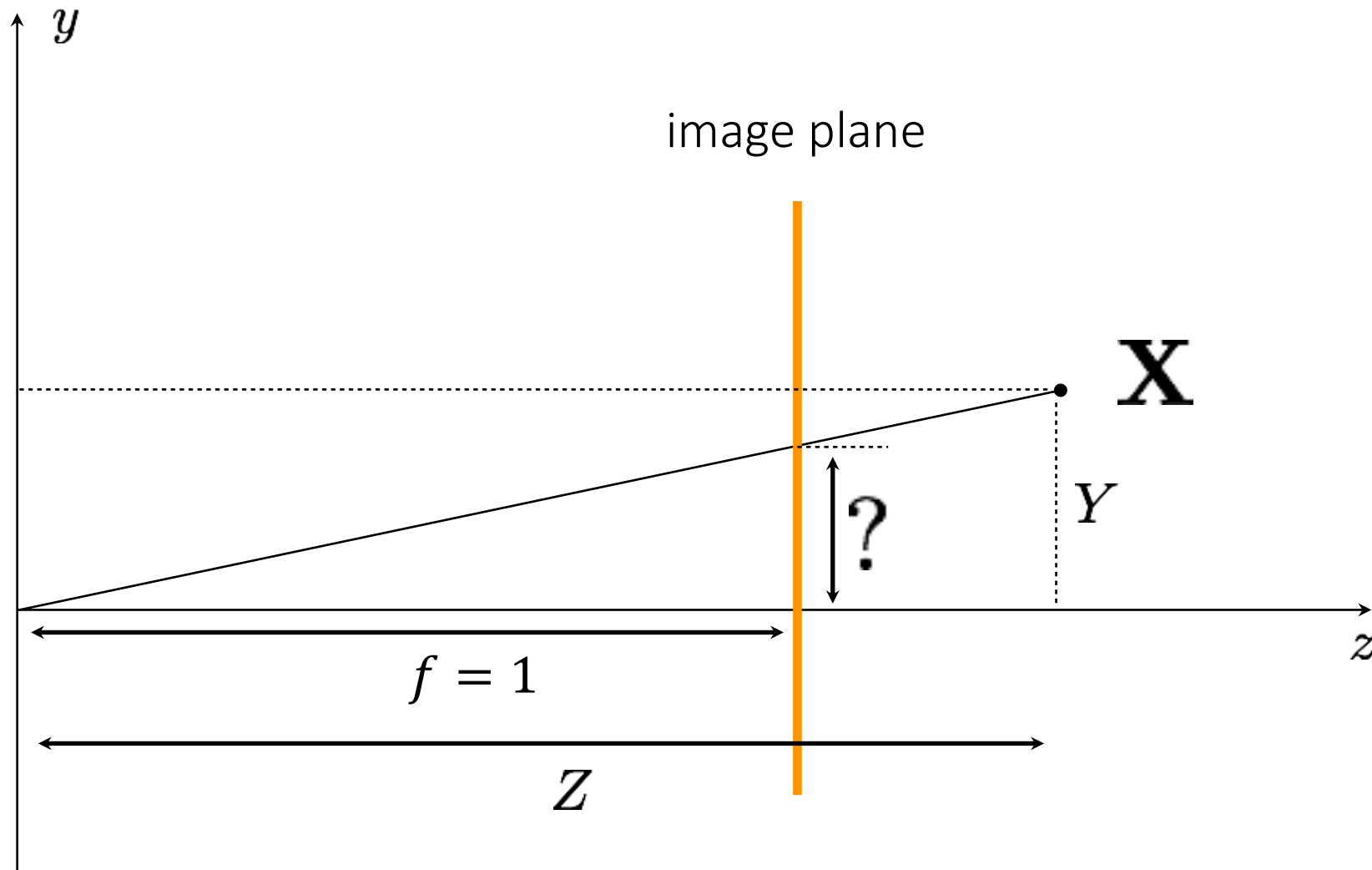


The (rearranged) pinhole camera



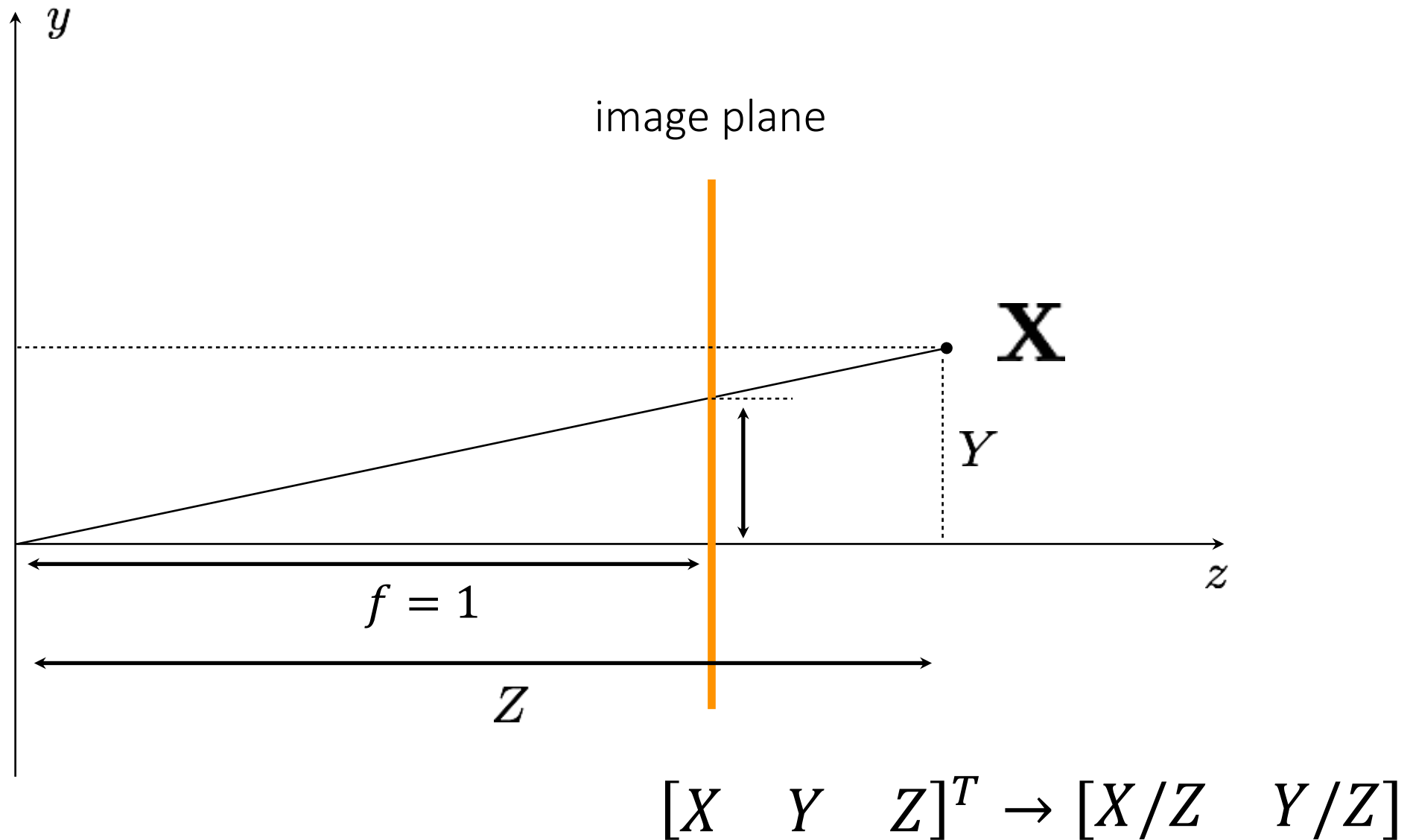
What is the equation for image coordinate \mathbf{x} in terms of \mathbf{X} ?

The 2D view of the (rearranged) pinhole camera

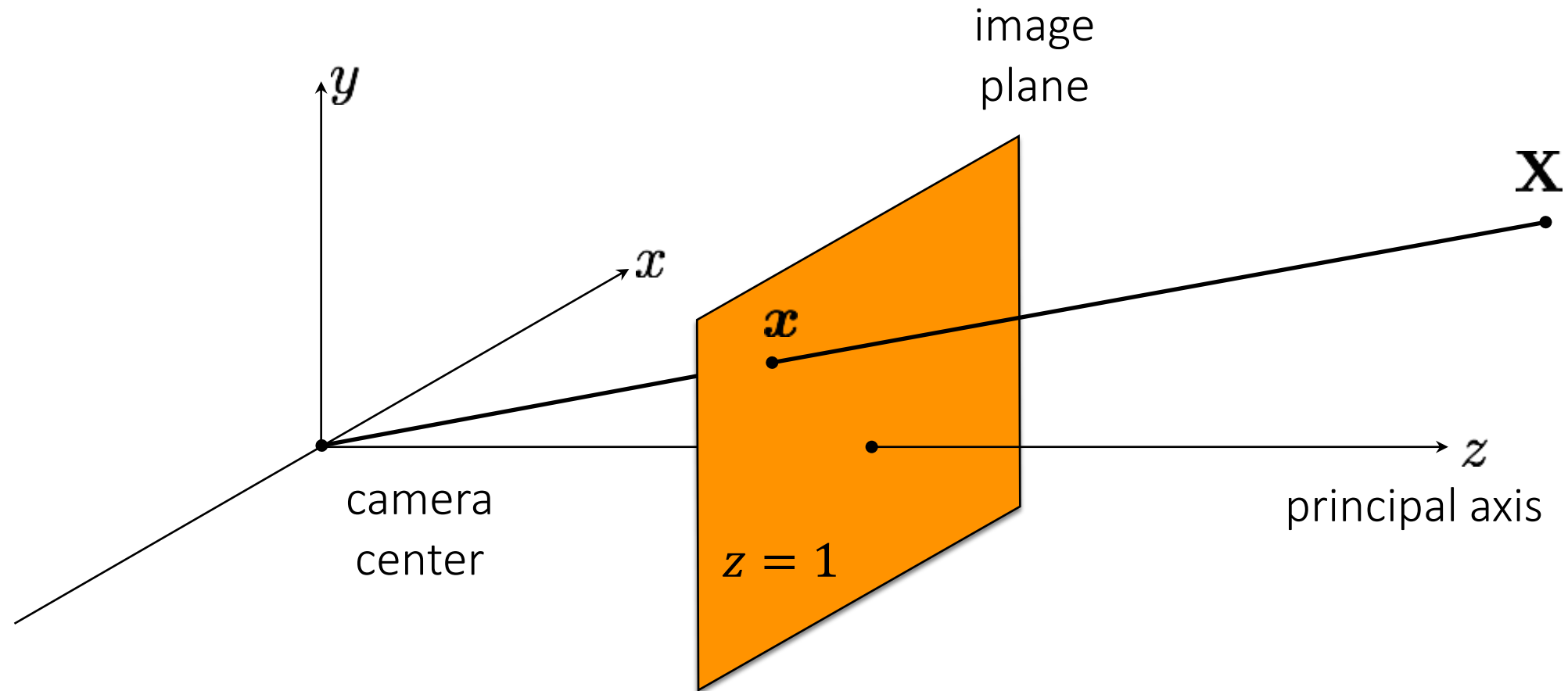


What is the equation for image coordinate x in terms of X ?

The 2D view of the (rearranged) pinhole camera



The (rearranged) pinhole camera



What is the camera matrix \mathbf{P} for a pinhole camera?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model in *homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

The perspective
projection matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

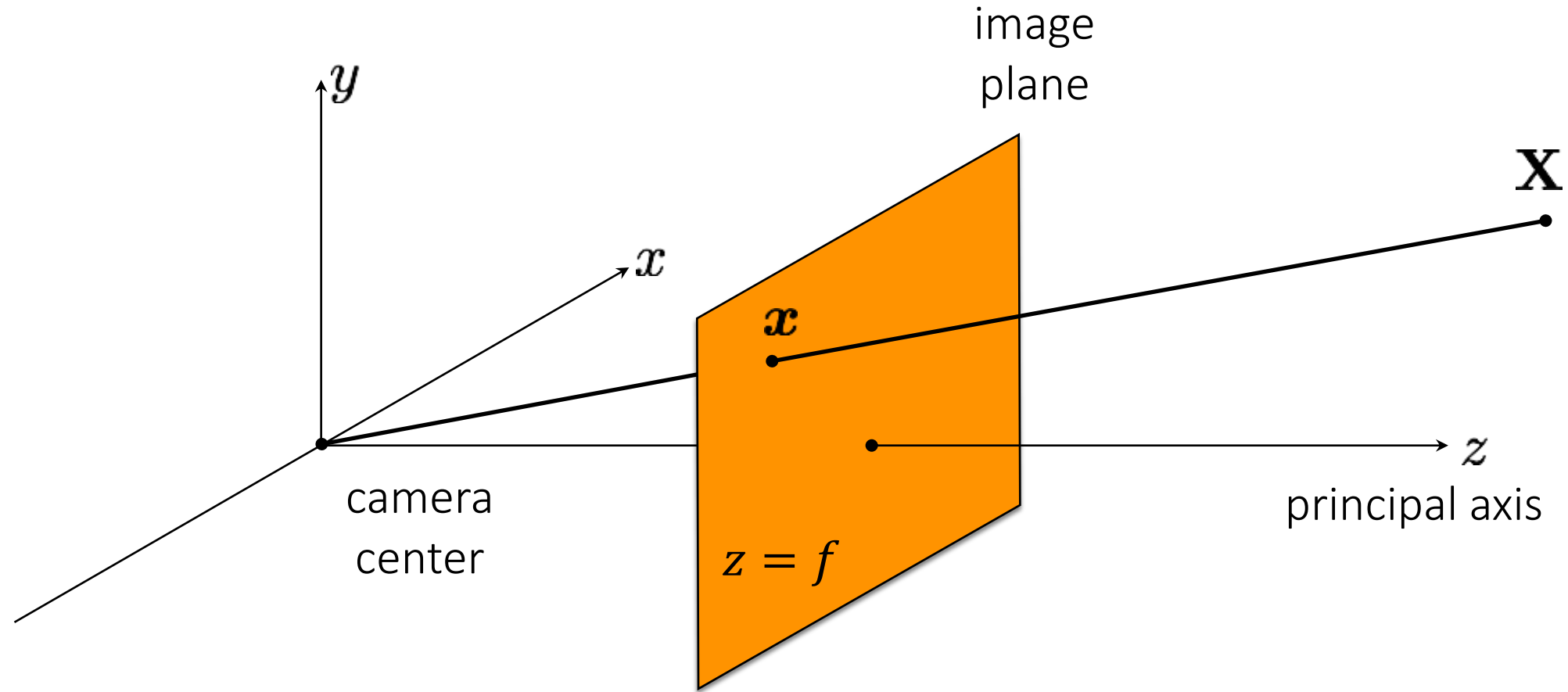
What does the pinhole camera projection look like?

The perspective
projection matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} = [\mathbf{I} \quad | \quad \mathbf{0}]$$

alternative way to write
the same thing

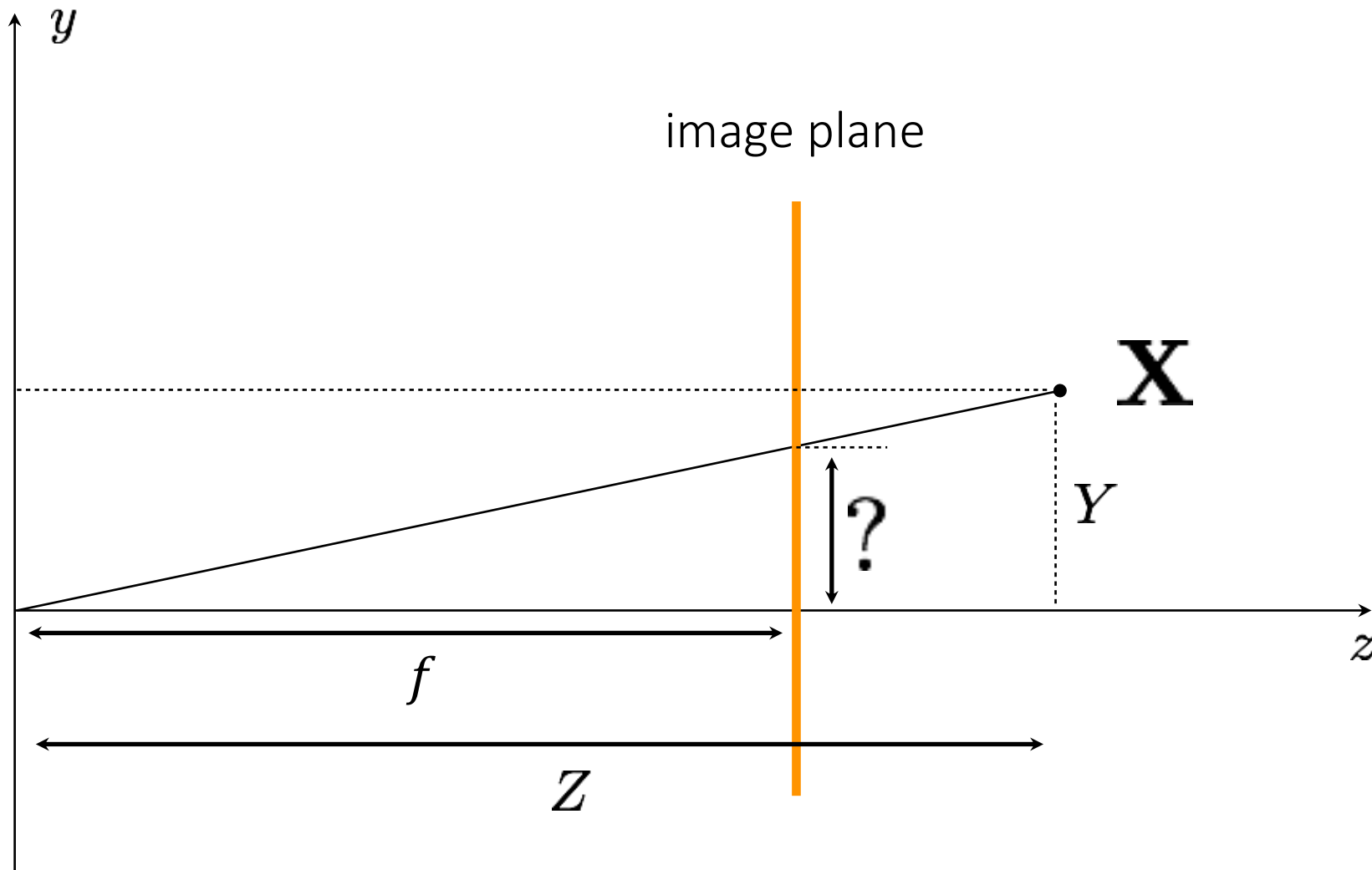
More general case: arbitrary focal length



What is the camera matrix \mathbf{P} for a pinhole camera?

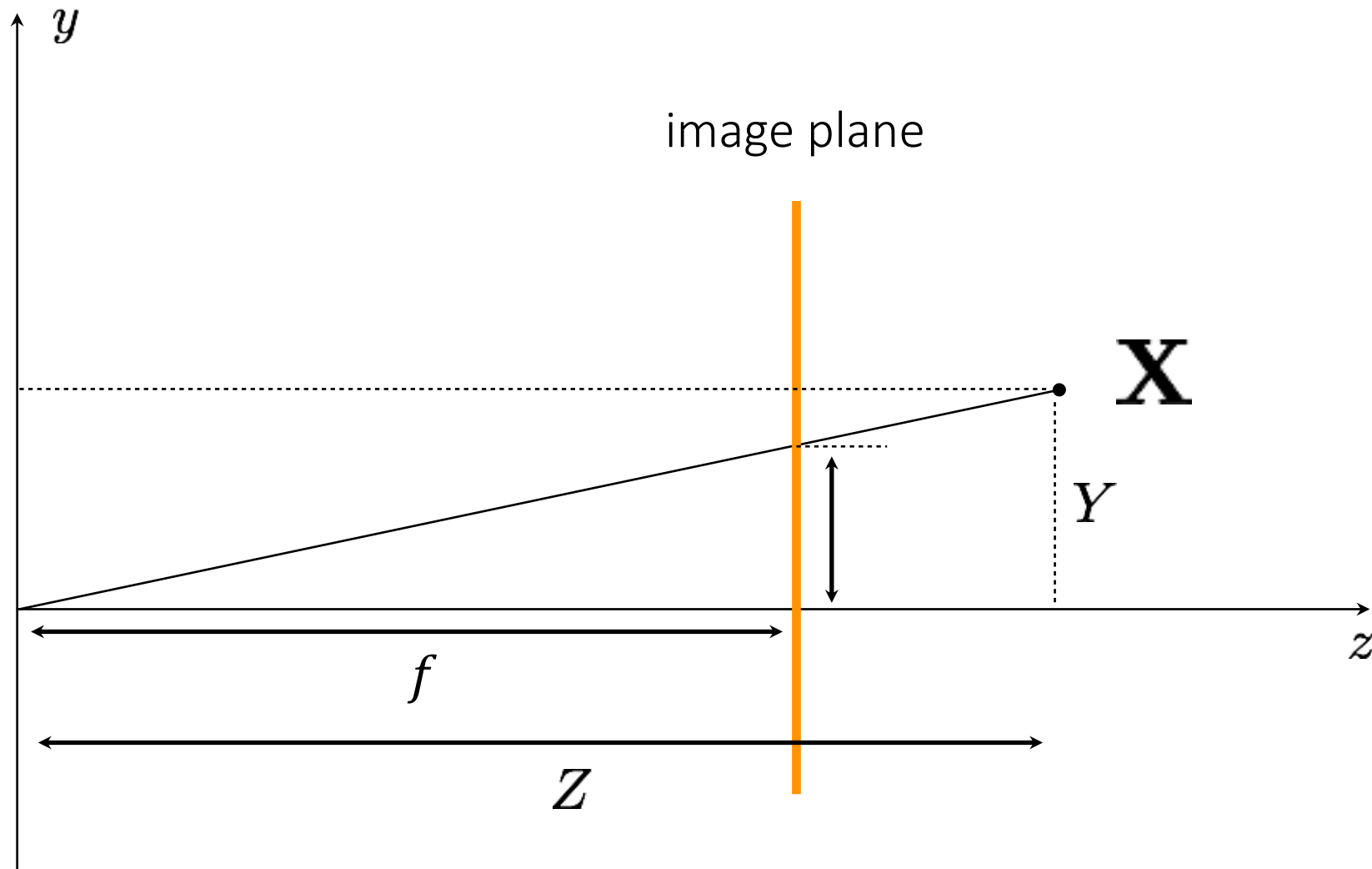
$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

More general (2D) case: arbitrary focal length



What is the equation for image coordinate x in terms of X ?

More general (2D) case: arbitrary focal length



$$[X \ Y \ Z]^T \mapsto [fX/Z \ fY/Z]^T$$

The pinhole camera matrix for arbitrary focal length

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \mapsto [fX/Z \quad fY/Z]^T$$

General camera model *in homogeneous coordinates*:

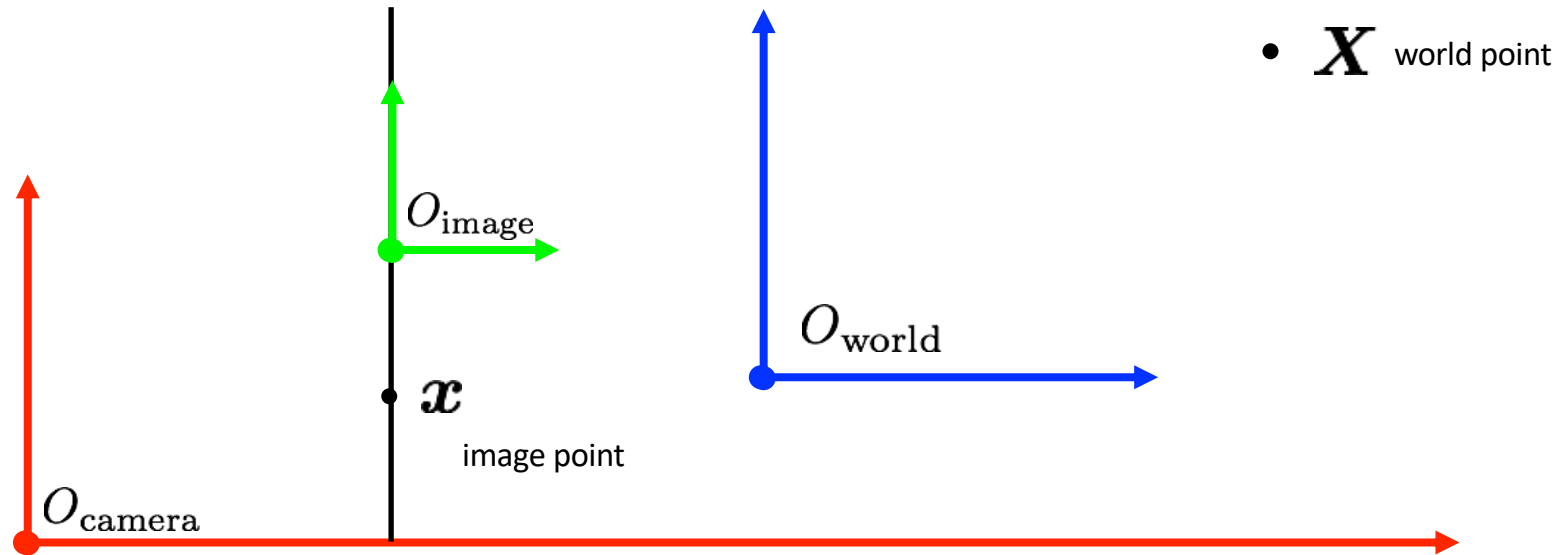
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

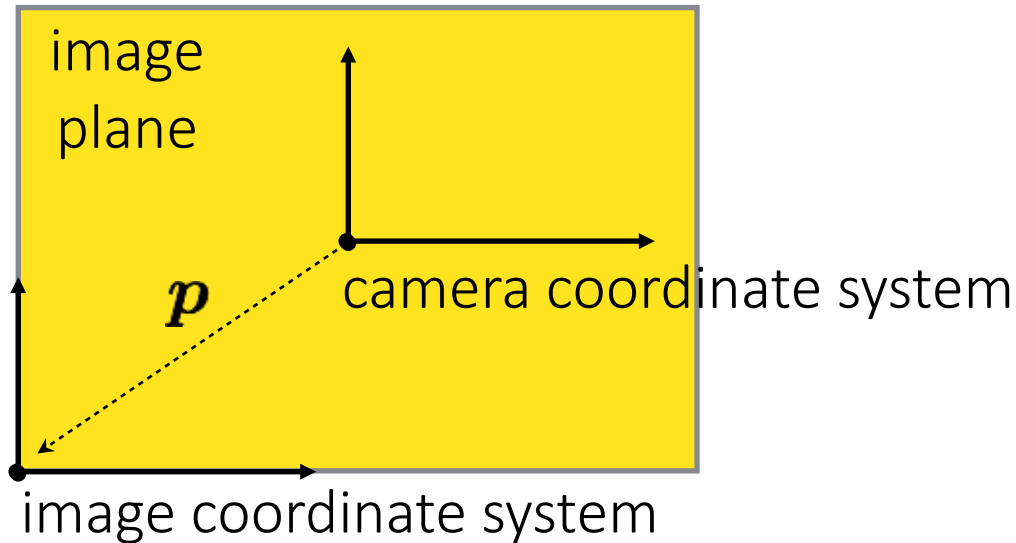
Generalizing the camera matrix

In general, the camera and image have *different* coordinate systems.



Generalizing the camera matrix

In particular, the camera origin and image origin may be different:

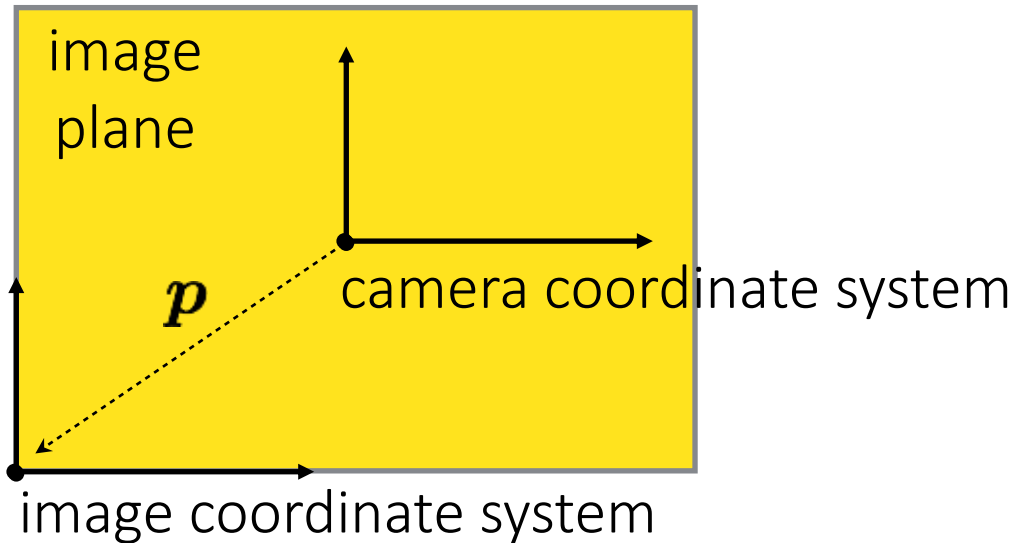


How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Generalizing the camera matrix

In particular, the camera origin and image origin may be different:



How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

shift vector
transforming
camera origin to
image origin

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

What does each part of the matrix represent?

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$



(homogeneous) transformation
from 2D to 2D, accounting for not
unit focal length and origin shift

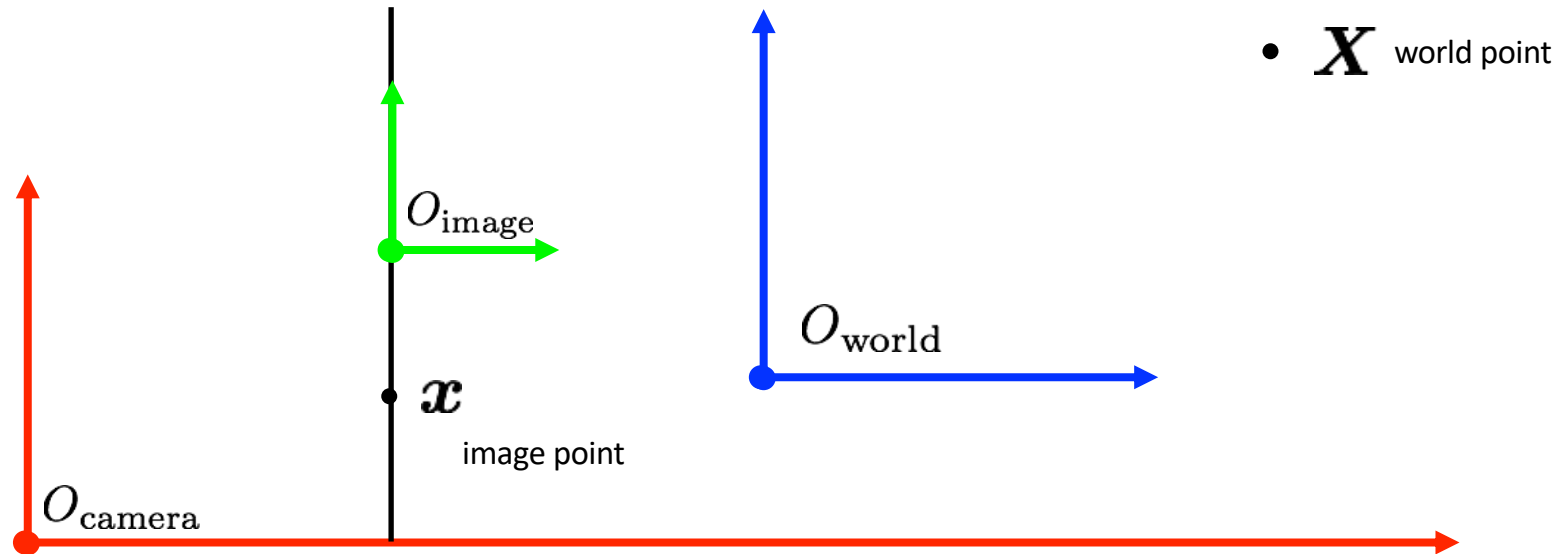


(homogeneous) perspective projection
from 3D to 2D, assuming image plane at
 $z = 1$ and shared camera/image origin

Also written as: $\mathbf{P} = \mathbf{K}[\mathbf{I}|\mathbf{0}]$ where $\mathbf{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$

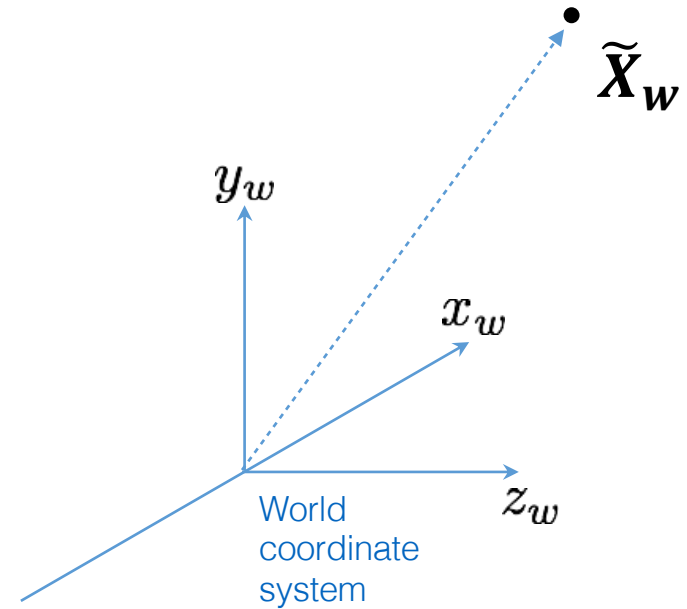
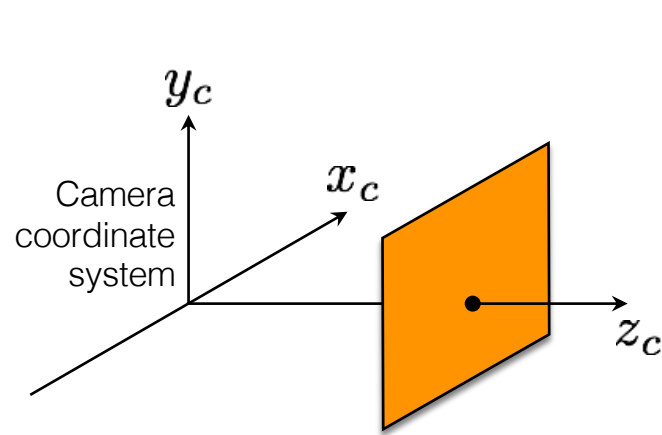
Generalizing the camera matrix

In general, there are *three*, generally different, coordinate systems.



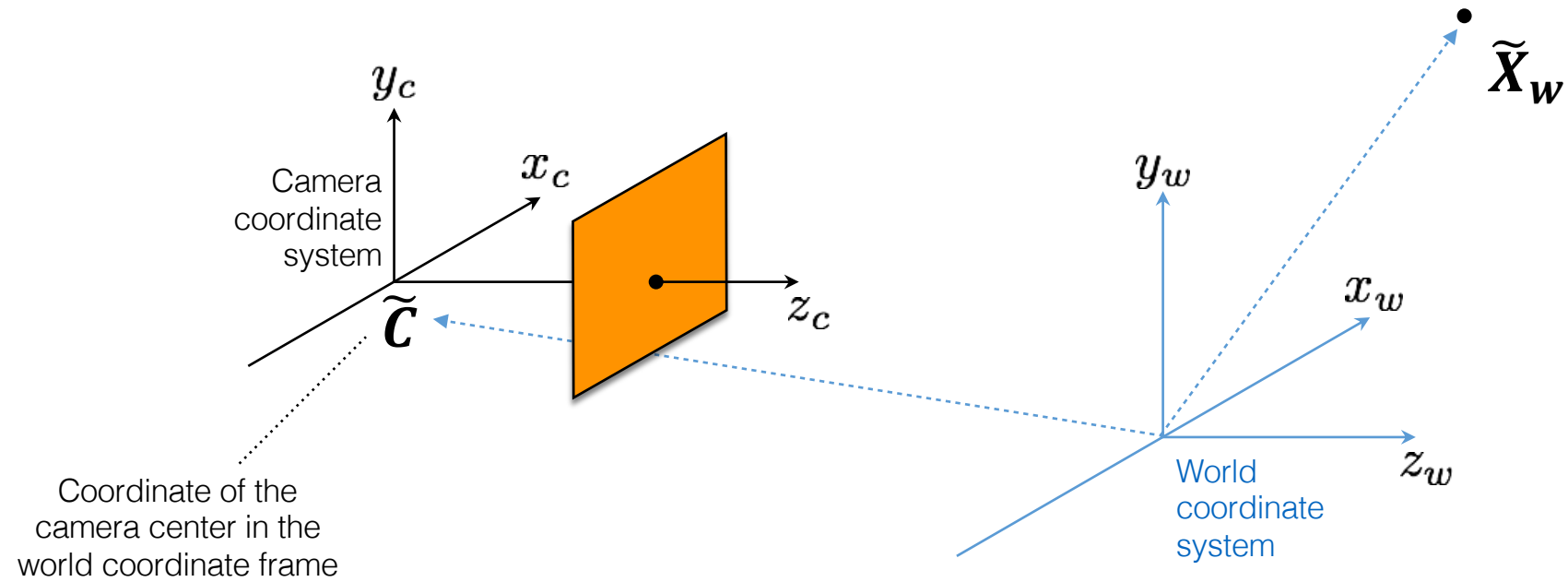
We need to know the transformations between them.

World-to-camera coordinate system transformation

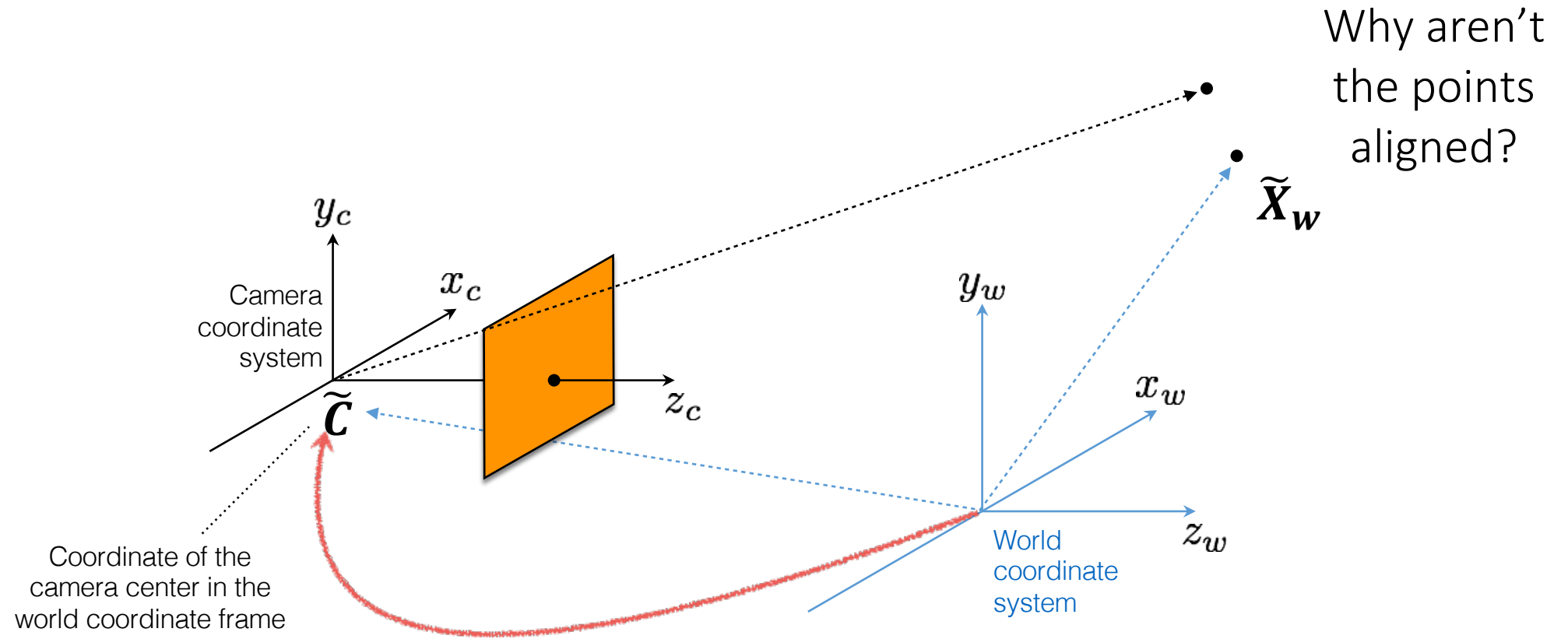


tilde means
heterogeneous
coordinates

World-to-camera coordinate system transformation



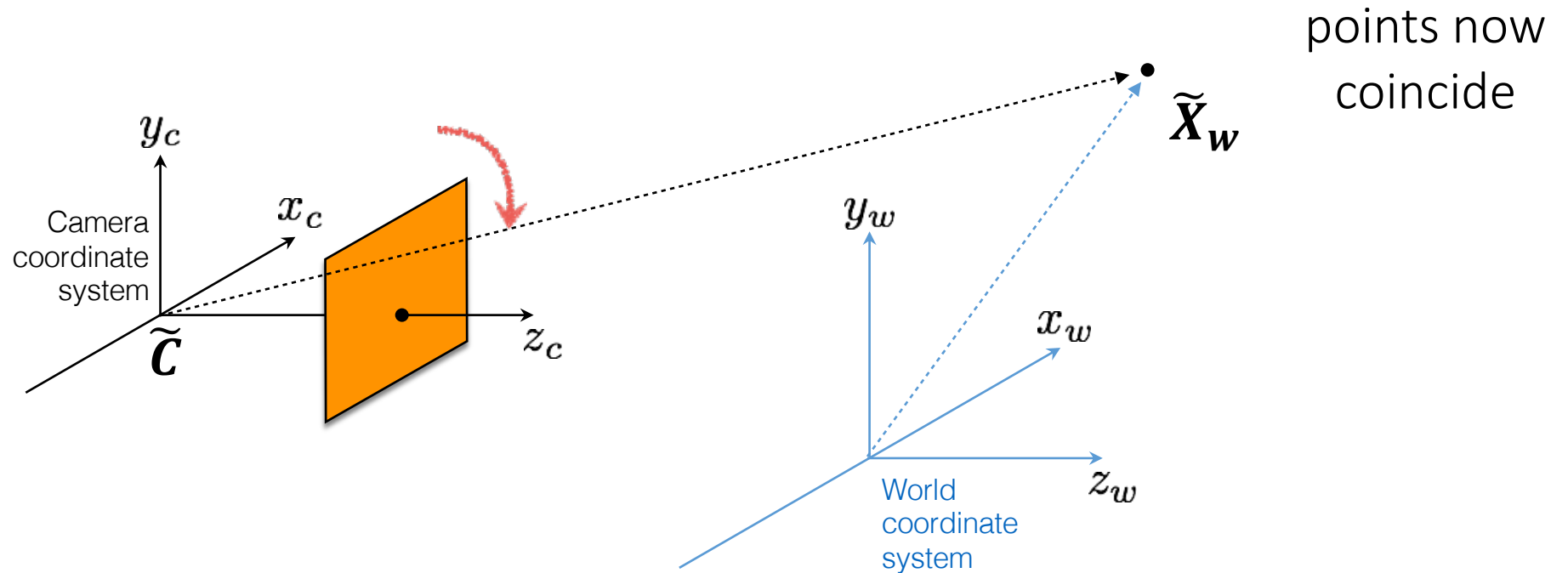
World-to-camera coordinate system transformation



$$(\tilde{X}_w - \tilde{C})$$

translate

World-to-camera coordinate system transformation



$$R \cdot (\tilde{X}_w - \tilde{C})$$

rotate

translate

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\tilde{\mathbf{X}}_c = \mathbf{R} \cdot (\tilde{\mathbf{X}}_w - \tilde{\mathbf{C}})$$

How do we write this transformation in homogeneous coordinates?

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\tilde{\mathbf{X}}_c = \mathbf{R} \cdot (\tilde{\mathbf{X}}_w - \tilde{\mathbf{C}})$$

In homogeneous coordinates, we have:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{X}_c = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}_w$$

Incorporating the transform in the camera matrix

The previous camera matrix is for homogeneous 3D coordinates in camera coordinate system:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_c = \mathbf{K}[\mathbf{I}|\mathbf{0}]\mathbf{X}_c$$

We also just derived:

$$\mathbf{X}_c = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}_w$$

Putting it all together

We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_w$$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix}$$

intrinsic parameters (3 x 3):
correspond to camera
internals (image-to-image
transformation)

perspective projection (3 x 4):
maps 3D to 2D points
(camera-to-image
transformation)

extrinsic parameters (4 x 4):
correspond to camera
externals (world-to-camera
transformation)

Putting it all together


We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_w$$


The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \quad \vdots \quad -\mathbf{RC} \right]$$

intrinsic parameters (3 x 3):
correspond to camera internals
(sensor not at $f = 1$ and origin shift)



extrinsic parameters (3 x 4):
correspond to camera externals
(world-to-image transformation)



General pinhole camera matrix

We can decompose the camera matrix like this:

$$\mathbf{P} = \mathbf{KR}[\mathbf{I} | -\mathbf{C}]$$

(translate first then rotate)

Another way to write the mapping:

$$\mathbf{P} = \mathbf{K}[\mathbf{R} | \mathbf{t}]$$

where $\mathbf{t} = -\mathbf{RC}$

(rotate first then translate)

General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & t_1 \\ r_4 & r_5 & r_6 & \cdots & t_2 \\ r_7 & r_8 & r_9 & \cdots & t_3 \end{bmatrix}$$

intrinsic
parameters

extrinsic
parameters

$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

3D rotation

3D translation

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}]$$

The diagram shows the equation $\mathbf{P} = \mathbf{KR}[\mathbf{I} \mid -\mathbf{C}]$ with four arrows pointing upwards from question marks to the terms \mathbf{P} , \mathbf{KR} , $[\mathbf{I} \mid -\mathbf{C}]$, and \mathbf{C} . The arrows originate from question marks located below the terms: one under \mathbf{P} , one under \mathbf{KR} , one under the vertical bar in $[\mathbf{I} \mid -\mathbf{C}]$, and one under \mathbf{C} .

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{KR}[\mathbf{I} | -\mathbf{C}]$$

The diagram shows the equation $\mathbf{P} = \mathbf{KR}[\mathbf{I} | -\mathbf{C}]$ with four arrows pointing from labels below to the terms \mathbf{P} , \mathbf{K} , \mathbf{I} , and \mathbf{C} . The label '3x3' is positioned above 'intrinsic' and below the arrow pointing to \mathbf{P} . The label 'intrinsic' is positioned below the arrow pointing to \mathbf{P} . Three question marks are positioned below the arrows pointing to \mathbf{K} , \mathbf{I} , and \mathbf{C} .

3x3
intrinsic

?

?

?

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{KR}[\mathbf{I} | \mathbf{C}]$$

3x3
intrinsics

3x3
3D rotation

?

?

Recap

What is the size and meaning of each term in the camera matrix?


$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | \mathbf{C}]$$

The diagram shows the equation $\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | \mathbf{C}]$ with four arrows pointing from labels below to the terms \mathbf{K} , \mathbf{R} , $[\mathbf{I} |$, and $\mathbf{C}]$. The labels are: 3×3 intrinsics, 3×3 3D rotation, 3×3 identity, and a question mark.

3×3 intrinsics 3×3 3D rotation 3×3 identity ?

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} \mid \mathbf{C}]$$


3x3 3x3 3x3 3x1
intrinsic 3D rotation identity 3D translation

Quiz

The camera matrix relates what two quantities?

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$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

homogeneous 3D points to 2D image points

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The camera matrix can be decomposed into?

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

intrinsic and extrinsic parameters

More general camera matrices

The following is the standard camera matrix we saw.

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \quad \left[\mathbf{R} \mid -\mathbf{RC} \right]$$

More general camera matrices

CCD camera: pixels may not be square.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \mid -\mathbf{RC} \right]$$

How many degrees of freedom?

More general camera matrices

CCD camera: pixels may not be square.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \mid -\mathbf{RC} \right]$$

How many degrees of freedom?

10 DOF

More general camera matrices

Finite projective camera: sensor be skewed.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \mid -\mathbf{RC} \right]$$

How many degrees of freedom?

More general camera matrices

Finite projective camera: sensor be skewed.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \mid -\mathbf{RC} \right]$$

How many degrees of freedom?

11 DOF