# Photometric stereo

16-385 Computer Vision
Spring 2024, Lecture 23

http://16385.courses.cs.cmu.edu/

# Overview of today’s lecture

- Some notes about radiometry.
- Quick overview of the n-dot-l model.
- Photometric stereo.
- Uncalibrated photometric stereo.
- Generalized bas-relief ambiguity.
- Shape from shading.

# Slide credits

Many of these slides were adapted from:

- Srinivasa Narasimhan (16-385, Spring 2014).
- Todd Zickler (Harvard University).
- Steven Gortler (Harvard University).
- Kayvon Fatahalian (Stanford University; CMU 15-462, Fall 2015).

# Five important equations/integrals to remember

Flux measured by a sensor of area X and directional receptivity W:

$$\Phi(W, X) = \int_X \int_W L(\hat{\boldsymbol{\omega}}, x) \cos\theta d\boldsymbol{\omega} dA$$

Reflectance equation:

$$L^{\text{out}}(\hat{\boldsymbol{\omega}}) = \int_{\Omega_{\text{in}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) L^{\text{in}}(\hat{\boldsymbol{\omega}}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\boldsymbol{\omega}}_{\text{in}}$$

Radiance under directional lighting and Lambertian BRDF ("n-dot-l shading"):

$$L^{\text{out}} = a \hat{\mathbf{n}}^\top \vec{\boldsymbol{\ell}}$$

Conversion of a (hemi)-spherical integral to a surface integral:

$$\int_{H^2} L_i(\mathrm{p}, \omega', t) \cos\theta \, \mathrm{d}\omega' = \int_A L(\mathrm{p}' \to \mathrm{p}, t) \frac{\cos\theta \cos\theta'}{||\mathrm{p}' - \mathrm{p}||^2} \, \mathrm{d}A'$$

Computing (hemi)-spherical integrals:

$$d\omega = \frac{dA}{r^2} = \sin\theta \, d\theta \, d\phi$$ and $$\int d\omega = \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\phi$$

# Five important equations/integrals to remember

Flux measured by a sensor of area X and directional receptivity W:

$$\Phi(W, X) = \int_X \int_W L(\hat{\omega}, x) \cos\theta d\omega dA$$

Reflectance equation:

$$L^{\text{out}}(\hat{\omega}) = \int_{\Omega_{\text{in}}} f(\hat{\omega}_{\text{in}}, \hat{\omega}_{\text{out}}) L^{\text{in}}(\hat{\omega}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\omega}_{\text{in}}$$

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# Photometric stereo

# Even simpler: Directional lighting

- Assume that, over the observed region of interest, all source of incoming flux is from one direction

$$L(x, \omega, t, \lambda) \longrightarrow L(\omega, t, \lambda) \longrightarrow s(t, \lambda)\delta(\omega = \omega_o(t))$$

$$L(x, \omega) \longrightarrow L(\omega) \longrightarrow s\delta(\omega = \omega_o)$$

- Convenient representation

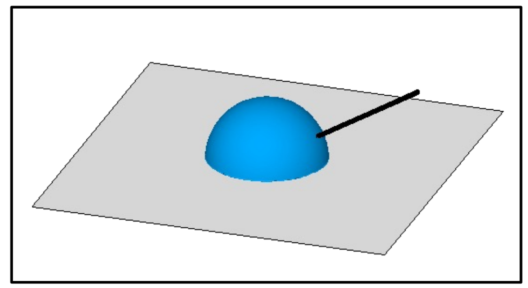
$$\vec{\boldsymbol{\ell}} = (\ell_x, \ell_y, \ell_z)$$

"light direction" $\hat{\boldsymbol{\ell}} = \dfrac{\vec{\boldsymbol{\ell}}}{||\vec{\boldsymbol{\ell}}||}$

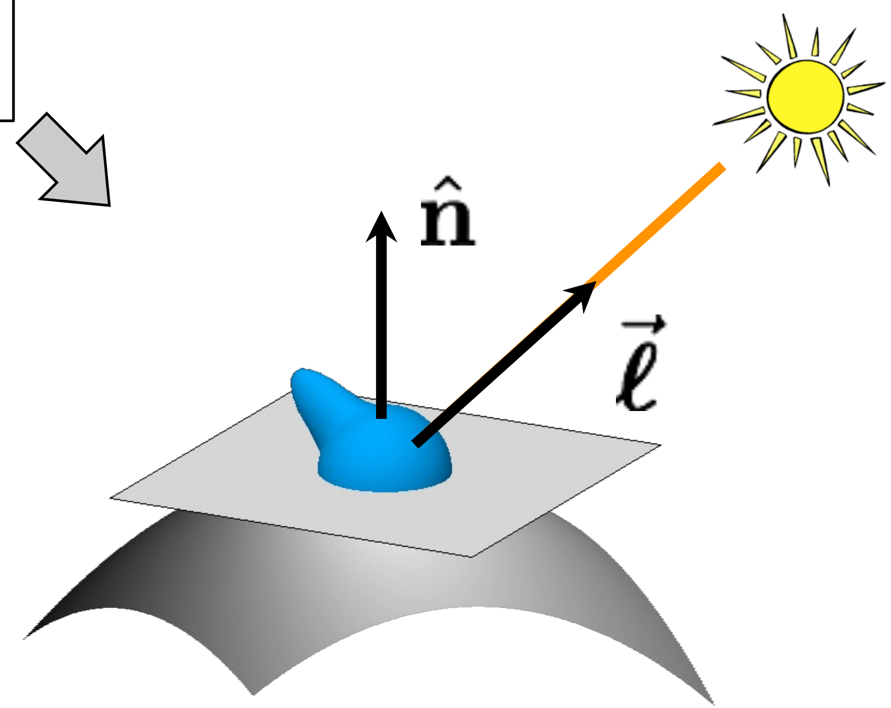
"light strength" $||\vec{\boldsymbol{\ell}}||$

# Simple shading

ASSUMPTION 1:
LAMBERTIAN



ASSUMPTION 2:
DIRECTIONAL LIGHTING

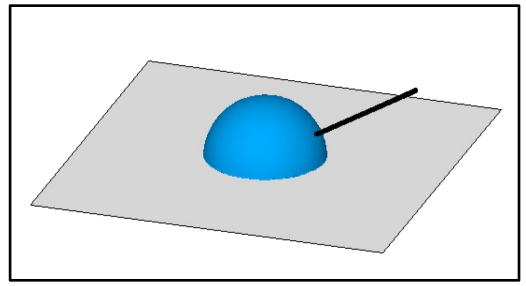



$$L^{\text{out}}(\hat{\boldsymbol{\omega}}) = \int_{\Omega_{\text{in}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) L^{\text{in}}(\hat{\boldsymbol{\omega}}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\boldsymbol{\omega}}_{\text{in}}$$

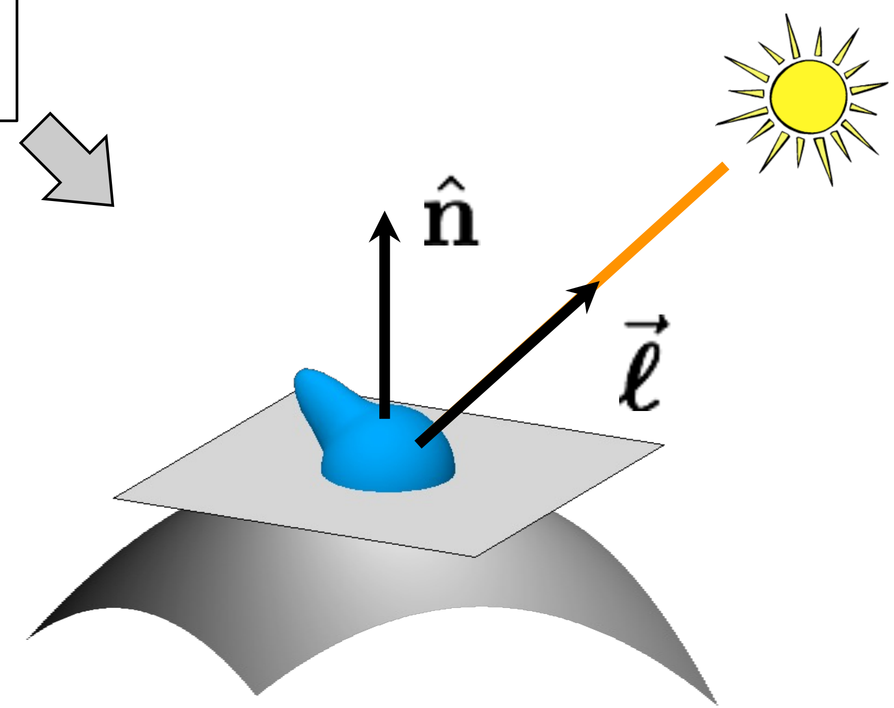
$$\boxed{I = a\hat{\mathbf{n}}^\top \vec{\boldsymbol{\ell}}}$$

# “N-dot-l” shading

ASSUMPTION 1:
LAMBERTIAN



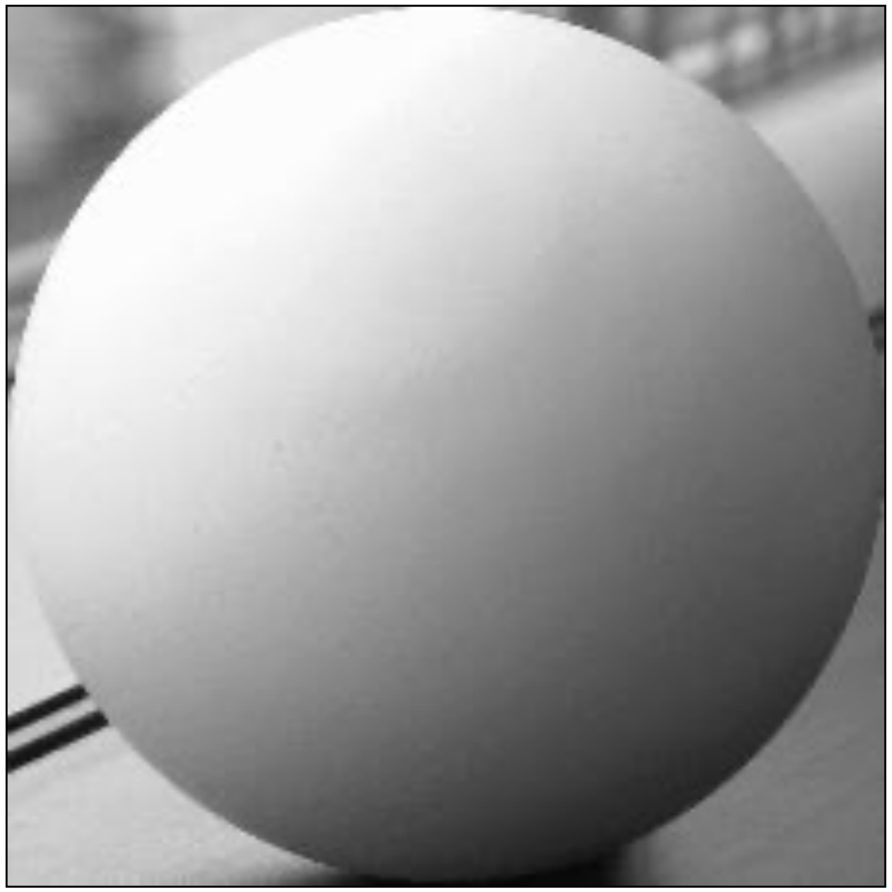
ASSUMPTION 2:
DIRECTIONAL LIGHTING



$$L^{\text{out}}(\hat{\boldsymbol{\omega}}) = \int_{\Omega_{\text{in}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) L^{\text{in}}(\hat{\boldsymbol{\omega}}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\boldsymbol{\omega}}_{\text{in}}$$

$$\boxed{I = a\hat{\mathbf{n}}^\top \vec{\boldsymbol{\ell}}}$$

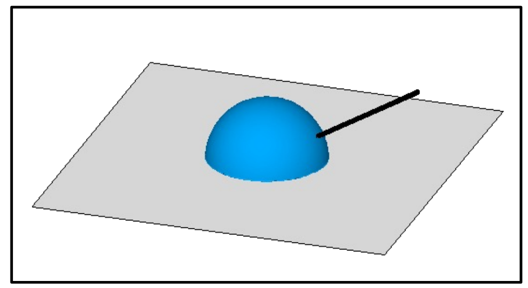
# Image Intensity and 3D Geometry



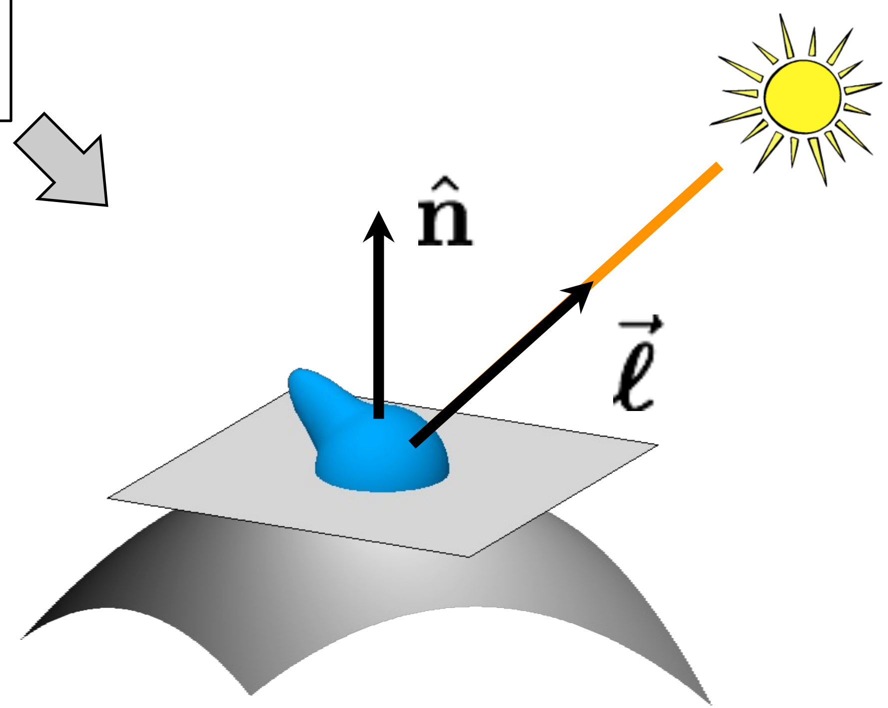
- *Shading* as a cue for shape reconstruction
- What is the relation between intensity and shape?

# "N-dot-l" shading

ASSUMPTION 1:
LAMBERTIAN



ASSUMPTION 2:
DIRECTIONAL LIGHTING

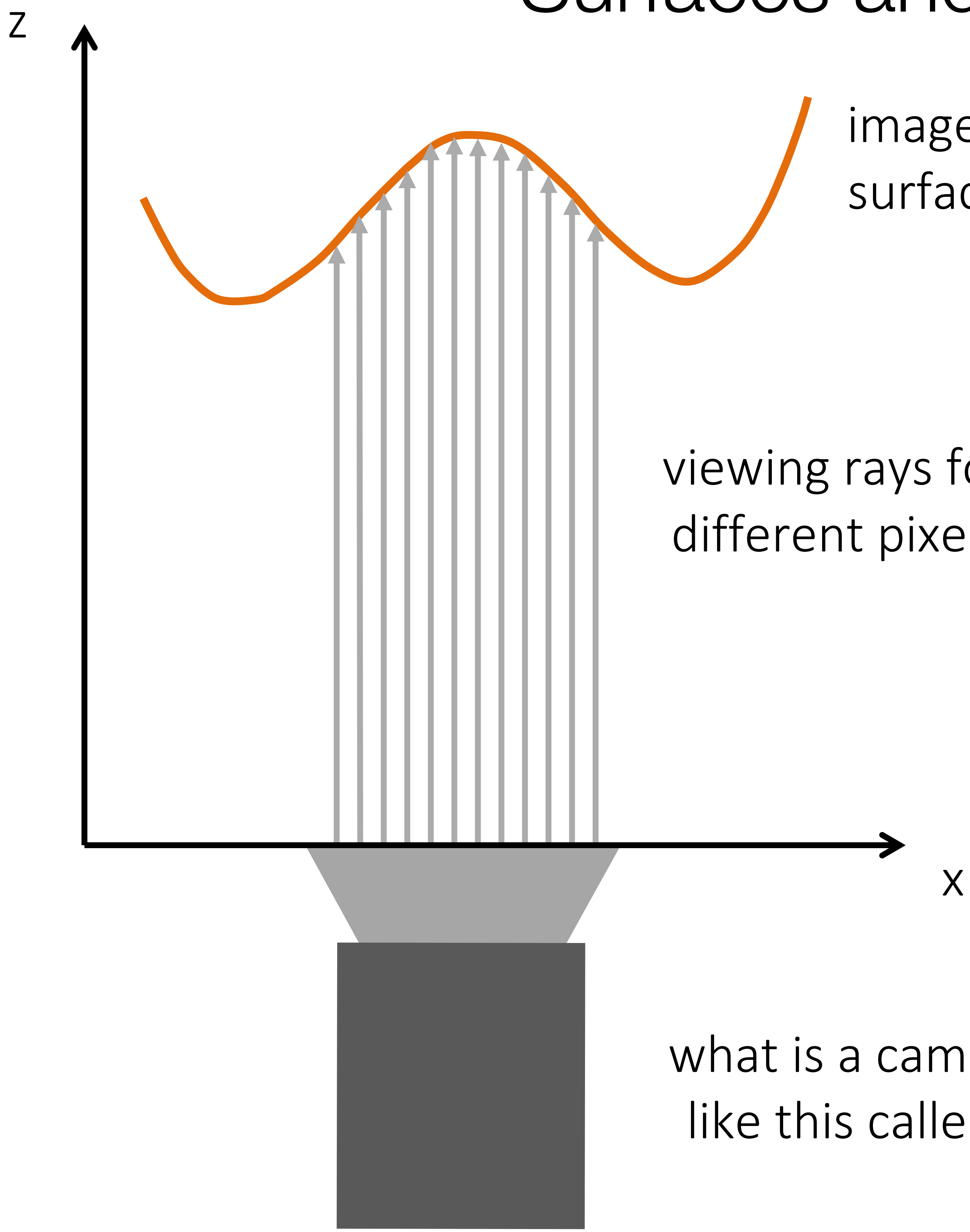



$$L^{\text{out}}(\hat{\boldsymbol{\omega}}) = \int_{\Omega_{\text{in}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) L^{\text{in}}(\hat{\boldsymbol{\omega}}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\boldsymbol{\omega}}_{\text{in}}$$

$$\boxed{I = a\hat{\mathbf{n}}^\top \vec{\boldsymbol{\ell}}}$$

Why do we call these normal "shape"?

# Surfaces and normals

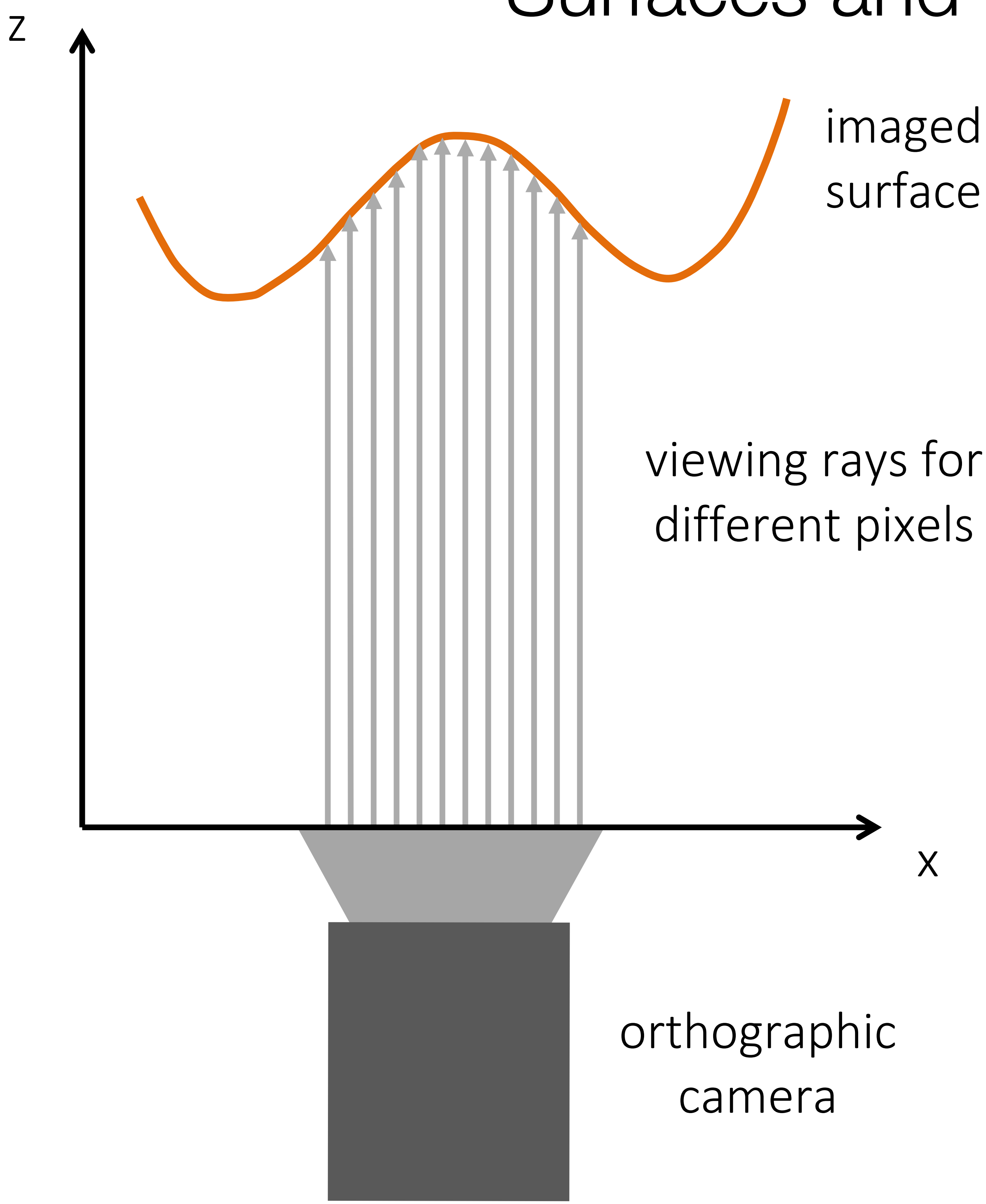



imaged surface

viewing rays for different pixels

what is a camera like this called?

# Surfaces and normals




Surface representation as a depth field (also known as Monge surface):

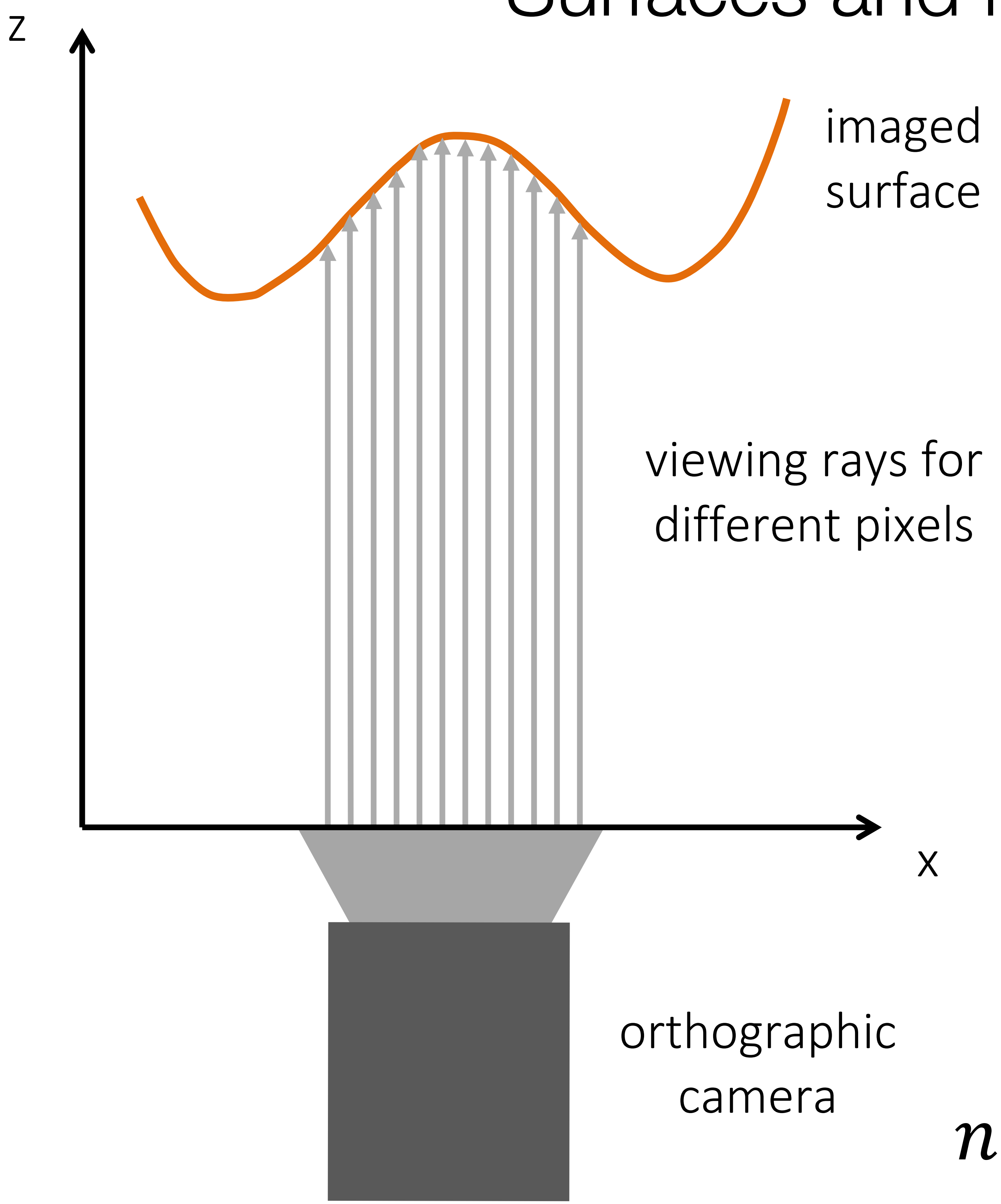
$$\underset{\uparrow}{z} = f(\underbrace{x, y})$$

pixel coordinates on image place

depth at each pixel

How does surface normal relate to this representation?

# Surfaces and normals




Surface representation as a depth image (also known as Monge surface):

$$z = f\underbrace{(x, y)}$$

pixel coordinates on image place

depth at each pixel

Unnormalized normal:

$$\tilde{n}(x,y) = \left(\frac{df}{dx}, \frac{df}{dy}, -1\right)$$

Actual normal:

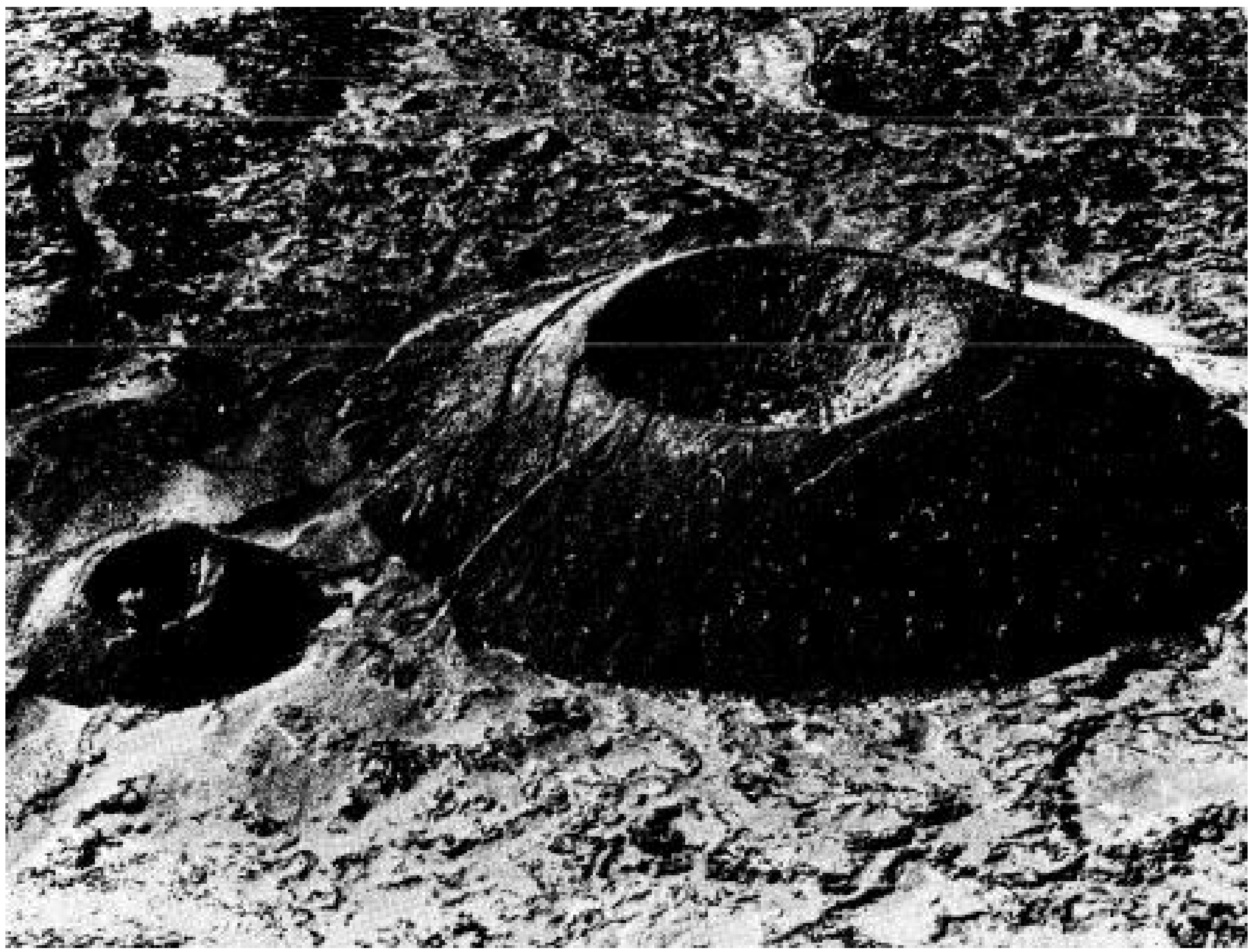
$$n(x,y) = \tilde{n}(x,y) / \|\tilde{n}(x,y)\|$$

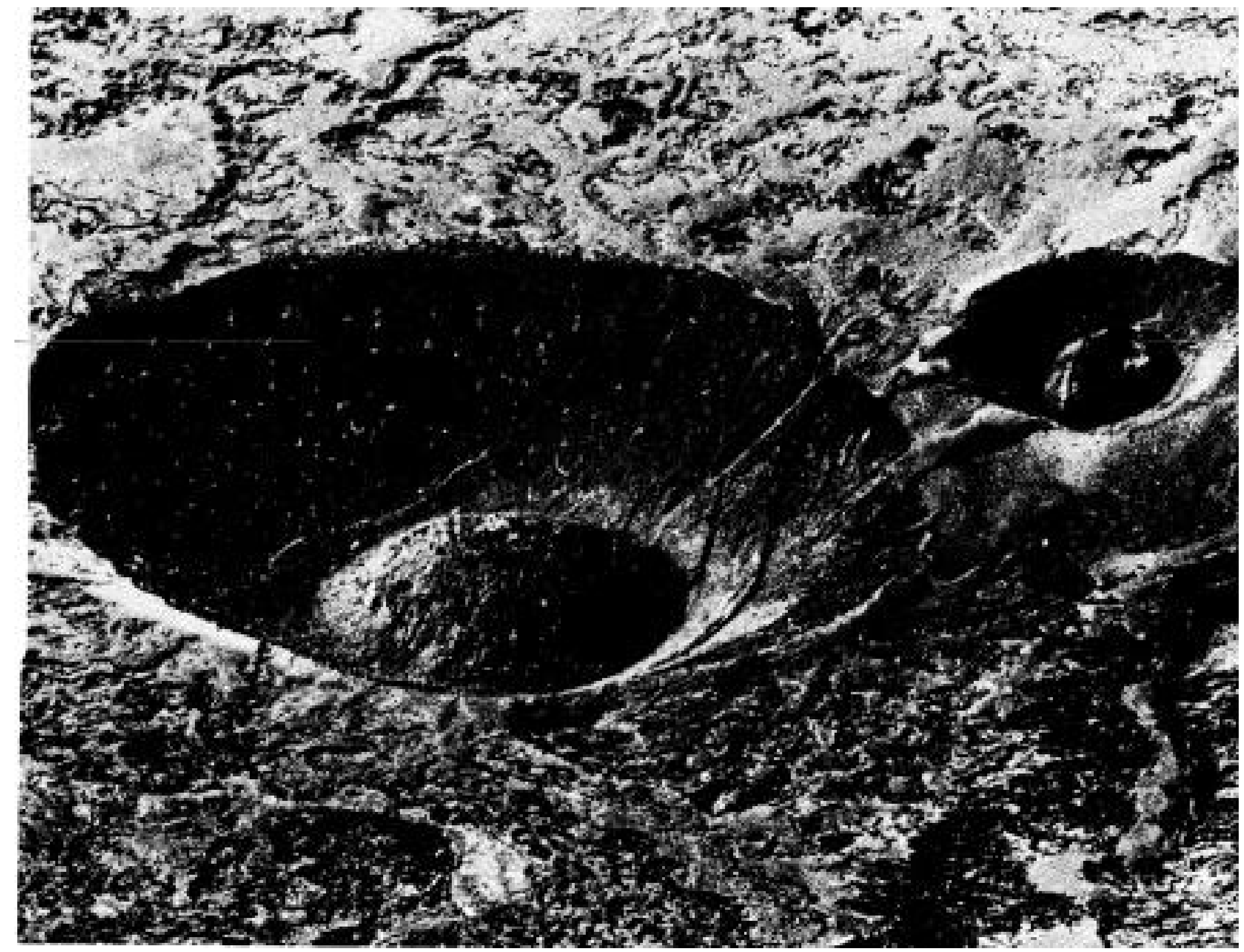
Normals are scaled spatial derivatives of depth image!

# Shape from a Single Image?

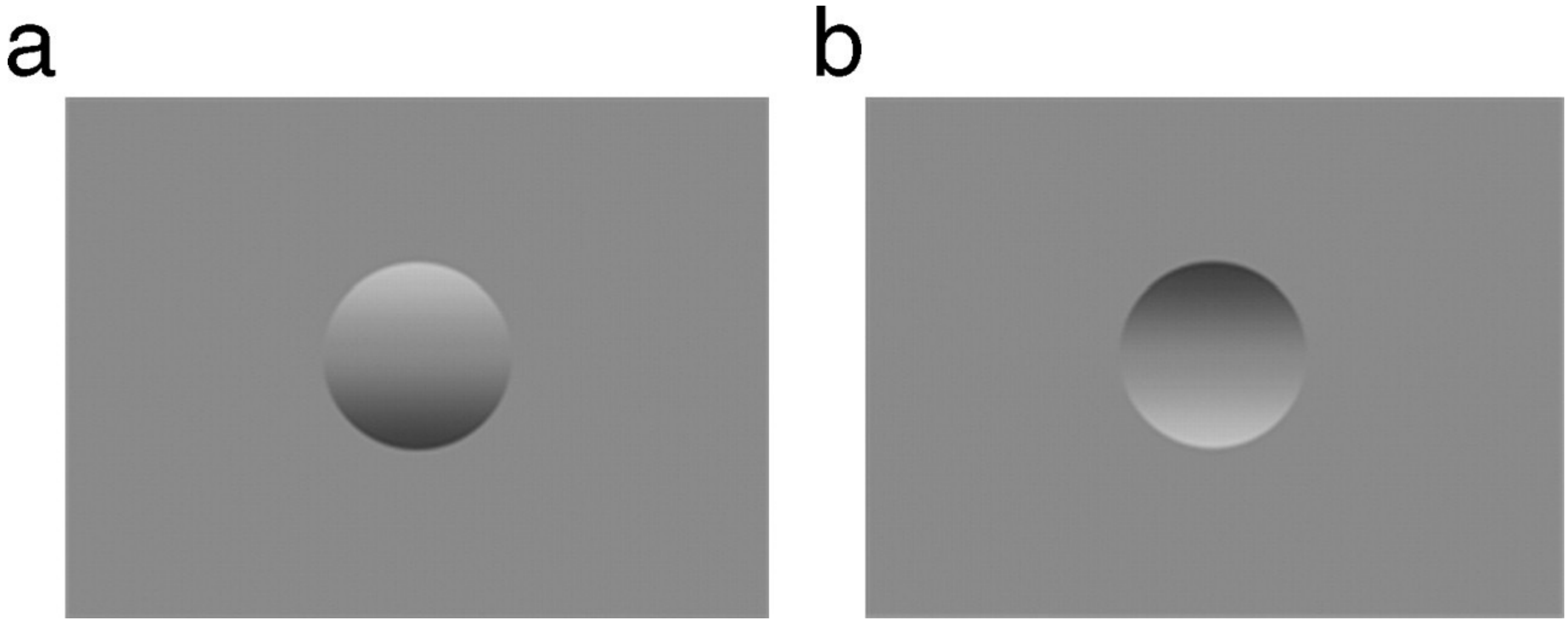
- Given a single image of an object with known surface reflectance taken under a known light source, can we recover the shape of the object?

# Human Perception





**Examples of the classic bump/dent stimuli used to test lighting assumptions when judging shape from shading, with shading orientations (a) 0° and (b) 180° from the vertical.**




Thomas R et al. J Vis 2010;10:6

Journal of VISION

# Human Perception

- Our brain often perceives shape from shading.
- Mostly, it makes many assumptions to do so.
- For example:

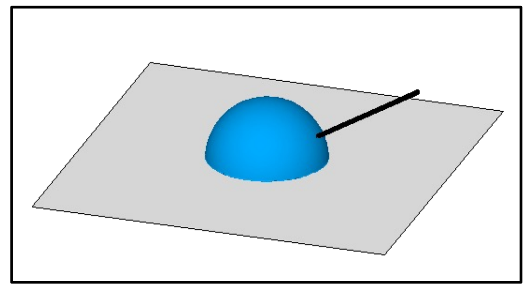
  Light is coming from above (sun).

  Biased by occluding contours.

by V. Ramachandran

# Single-lighting is ambiguous

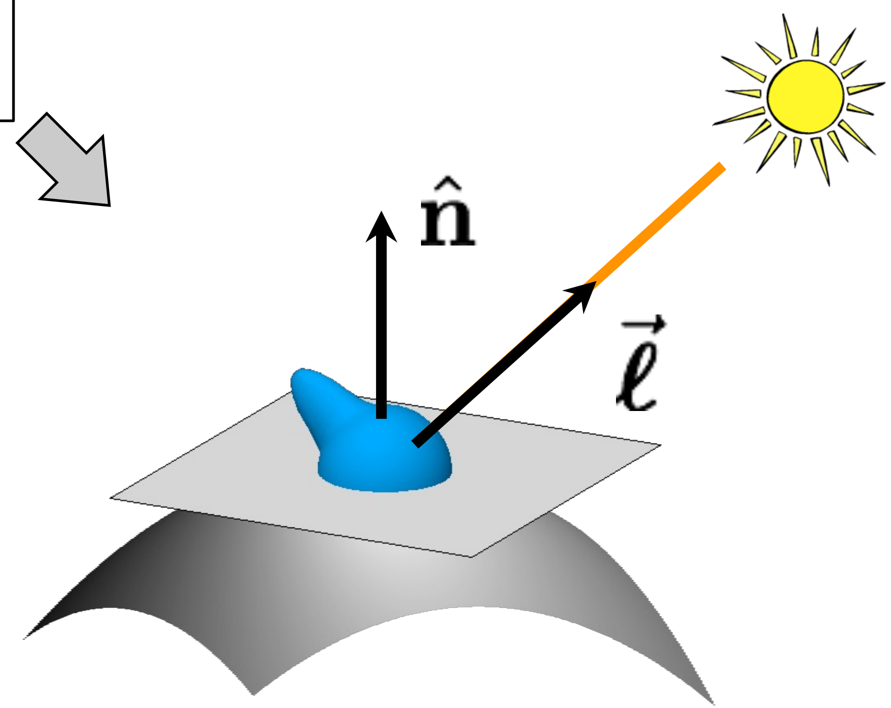
ASSUMPTION 1:
LAMBERTIAN



ASSUMPTION 2:
DIRECTIONAL LIGHTING

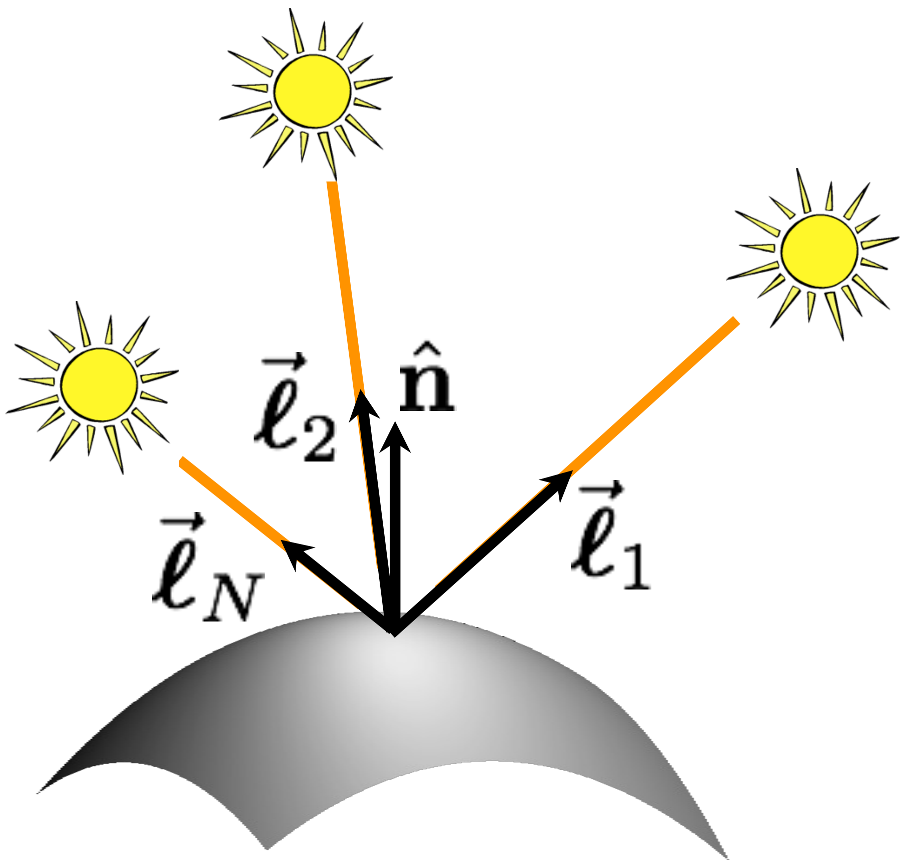
$$L^{\text{out}}(\hat{\boldsymbol{\omega}}) = \int_{\Omega_{\text{in}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) L^{\text{in}}(\hat{\boldsymbol{\omega}}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\boldsymbol{\omega}}_{\text{in}}$$

$$\boxed{I = a\hat{\mathbf{n}}^\top \vec{\boldsymbol{\ell}}}$$



# Lambertian photometric stereo

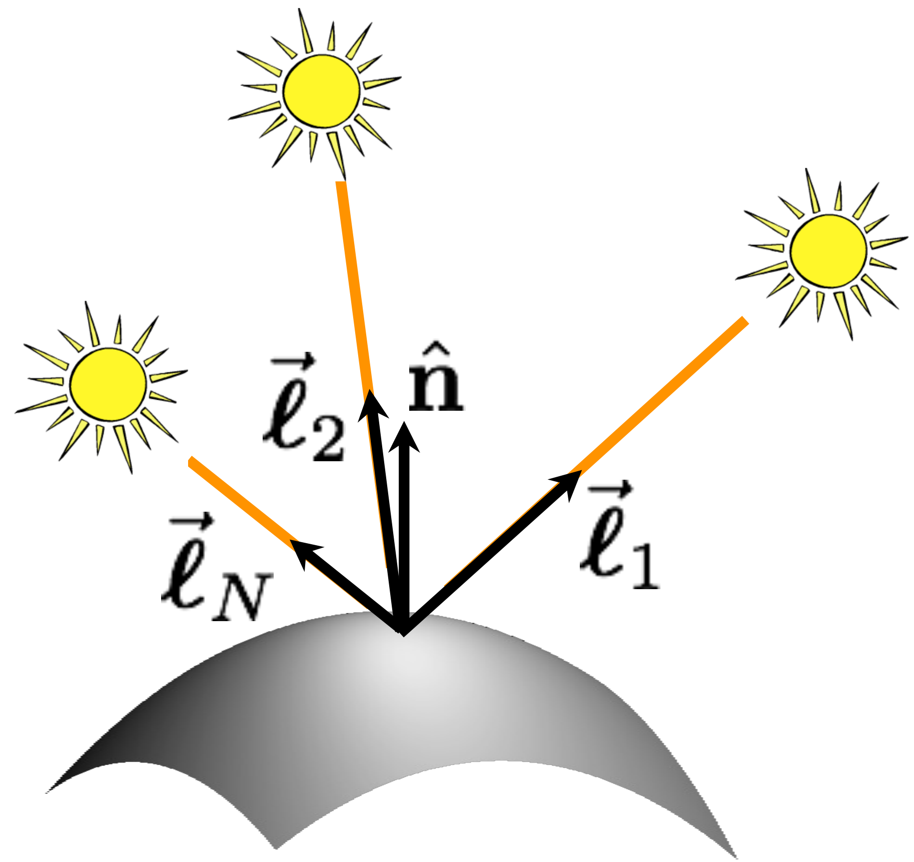
$$
\begin{gathered}
I_1 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_1 \\
I_2 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_2 \\
\vdots \\
I_N = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_N
\end{gathered}
$$




Assumption: We know the lighting directions.

# Lambertian photometric stereo

$$
\boxed{
\begin{aligned}
I_1 &= a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_1 \\
I_2 &= a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_2 \\
&\vdots \\
I_N &= a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_N
\end{aligned}
}
$$




define “pseudo-normal” $\vec{\boldsymbol{b}} \triangleq a\hat{\boldsymbol{n}}$

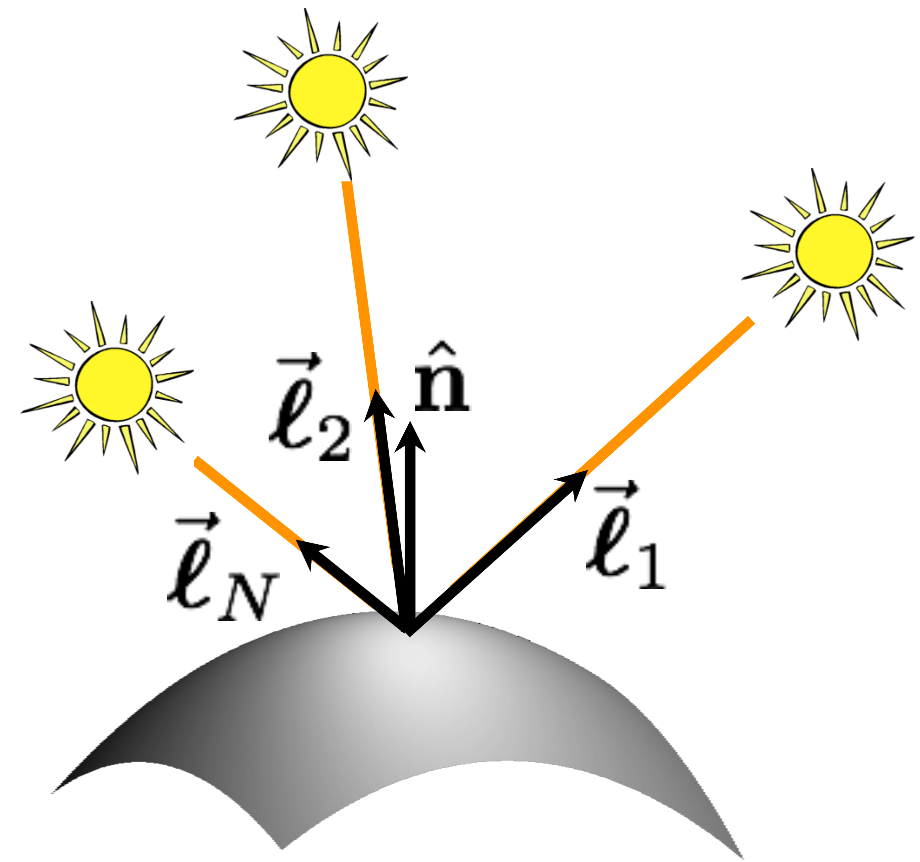
solve linear system for pseudo-normal

$$
\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{\ell}}_1^\top \\ \vec{\boldsymbol{\ell}}_2^\top \\ \vdots \\ \vec{\boldsymbol{\ell}}_N^\top \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{b}} \end{bmatrix}
$$

What are the dimensions of these matrices?

# Lambertian photometric stereo

$$I_1 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_1$$
$$I_2 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_2$$
$$\vdots$$
$$I_N = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_N$$

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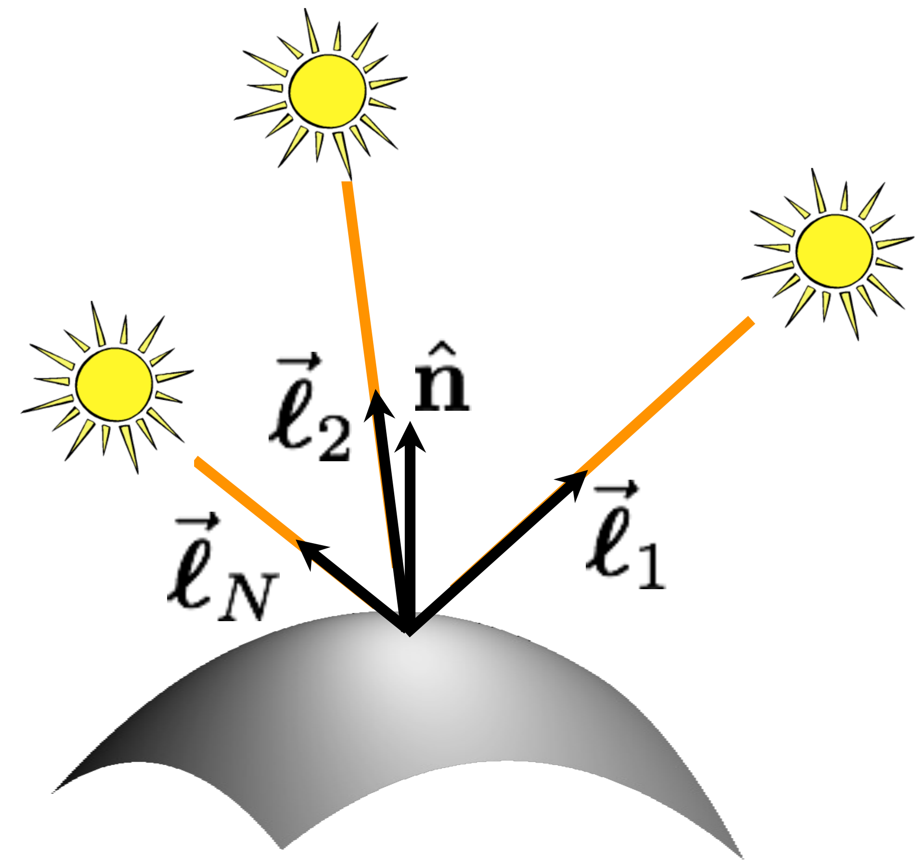
$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} \vec{\boldsymbol{\ell}}_1^\top \\ \vec{\boldsymbol{\ell}}_2^\top \\ \vdots \\ \vec{\boldsymbol{\ell}}_N^\top \end{bmatrix}_{N\times 3} \begin{bmatrix} \vec{\boldsymbol{b}} \end{bmatrix}_{3\times 1}$$

What are the knowns and unknowns?

# Lambertian photometric stereo

$$I_1 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_1$$
$$I_2 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_2$$
$$\vdots$$
$$I_N = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_N$$

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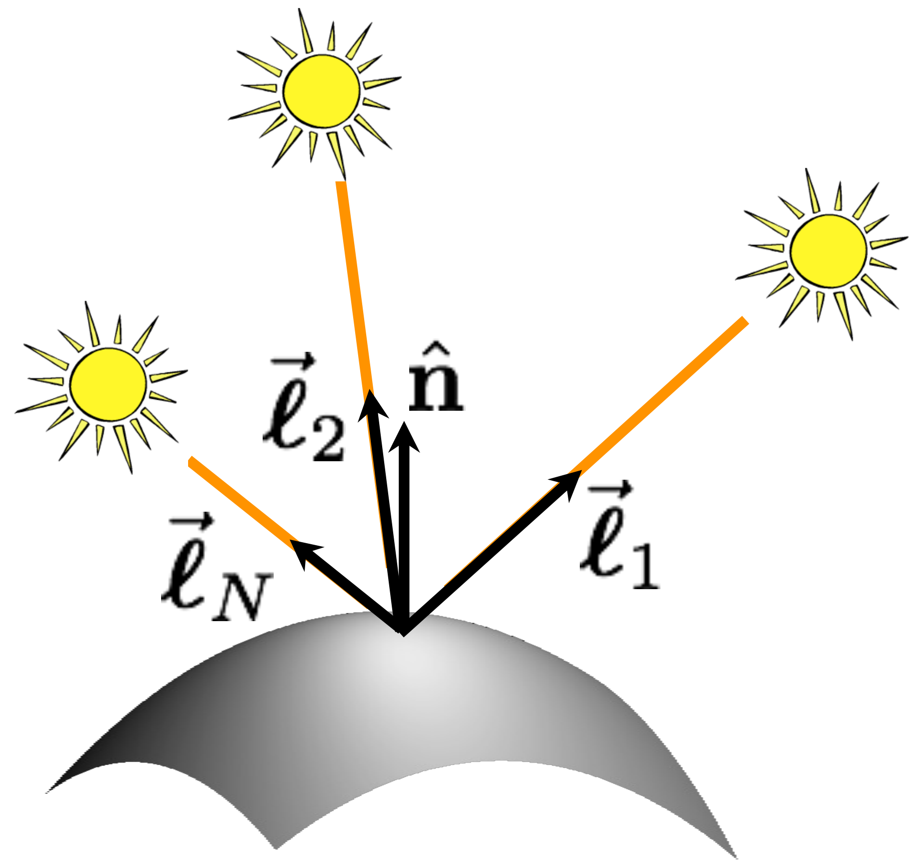
How many lights do I need for unique solution?

# Lambertian photometric stereo

$$\begin{aligned} I_1 &= a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_1 \\ I_2 &= a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_2 \\ &\vdots \\ I_N &= a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_N \end{aligned}$$

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once system is solved, b gives normal direction and albedo

How do we solve this system?

# Solving the Equation with three lights

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_2 \end{bmatrix}}_{\substack{\mathbf{I} \\ 3\times 1}} = \underbrace{\begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \mathbf{s}_3^T \end{bmatrix}}_{\substack{\mathbf{S} \\ 3\times 3}} \underbrace{\rho\mathbf{n}}_{\substack{\tilde{\mathbf{n}} \\ 3\times 1}}$$

$$\tilde{\mathbf{n}} = \mathbf{S}^{-1}\mathbf{I}$$

inverse

$$\rho = |\tilde{\mathbf{n}}|$$

$$\mathbf{n} = \frac{\tilde{\mathbf{n}}}{|\tilde{\mathbf{n}}|} = \frac{\tilde{\mathbf{n}}}{\rho}$$

Is there any reason to use more than three lights?

# More than Three Light Sources

- Get better SNR by using more lights

$$\begin{bmatrix} I_1 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_N^T \end{bmatrix} \rho \mathbf{n}$$

- Least squares solution:

$$\mathbf{I} = \mathbf{S}\tilde{\mathbf{n}} \longleftarrow \qquad N \times 1 = (\underline{N \times 3})(3 \times 1)$$

$$\mathbf{S}^T\mathbf{I} = \mathbf{S}^T\mathbf{S}\tilde{\mathbf{n}}$$

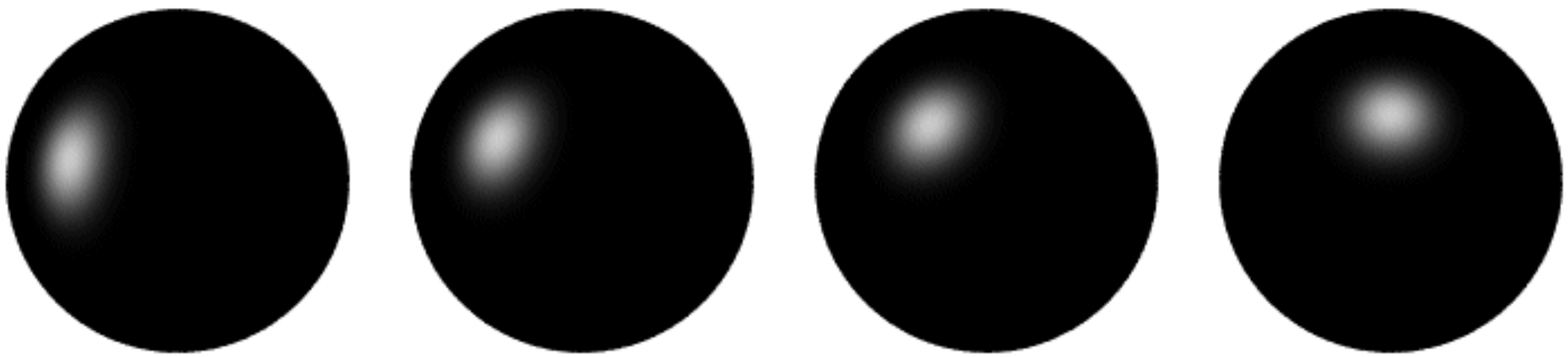
$$\tilde{\mathbf{n}} = \boxed{\left(\mathbf{S}^T\mathbf{S}\right)^{-1}\mathbf{S}^T}\mathbf{I}$$

Moore-Penrose pseudo inverse

- Solve for $\rho, \mathbf{n}$ as before

# Computing light source directions

- Trick: place a chrome sphere in the scene



  – the location of the highlight tells you the source direction

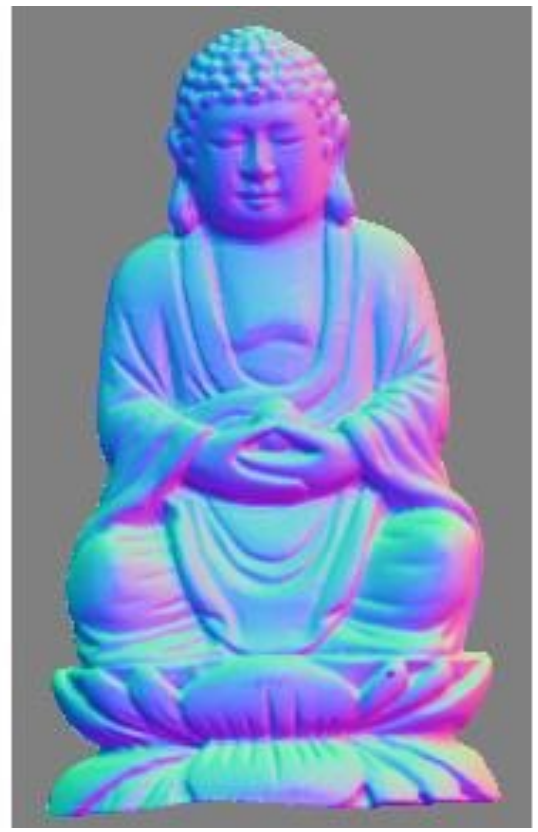
# Limitations

- Big problems
  - Doesn’t work for shiny things, semi-translucent things
  - Shadows, inter-reflections
- Smaller problems
  - Camera and lights have to be distant
  - Calibration requirements
    - measure light source directions, intensities
    - camera response function

# Depth from normals

- Solving the linear system per-pixel gives us an estimated surface normal for each pixel

Input photo

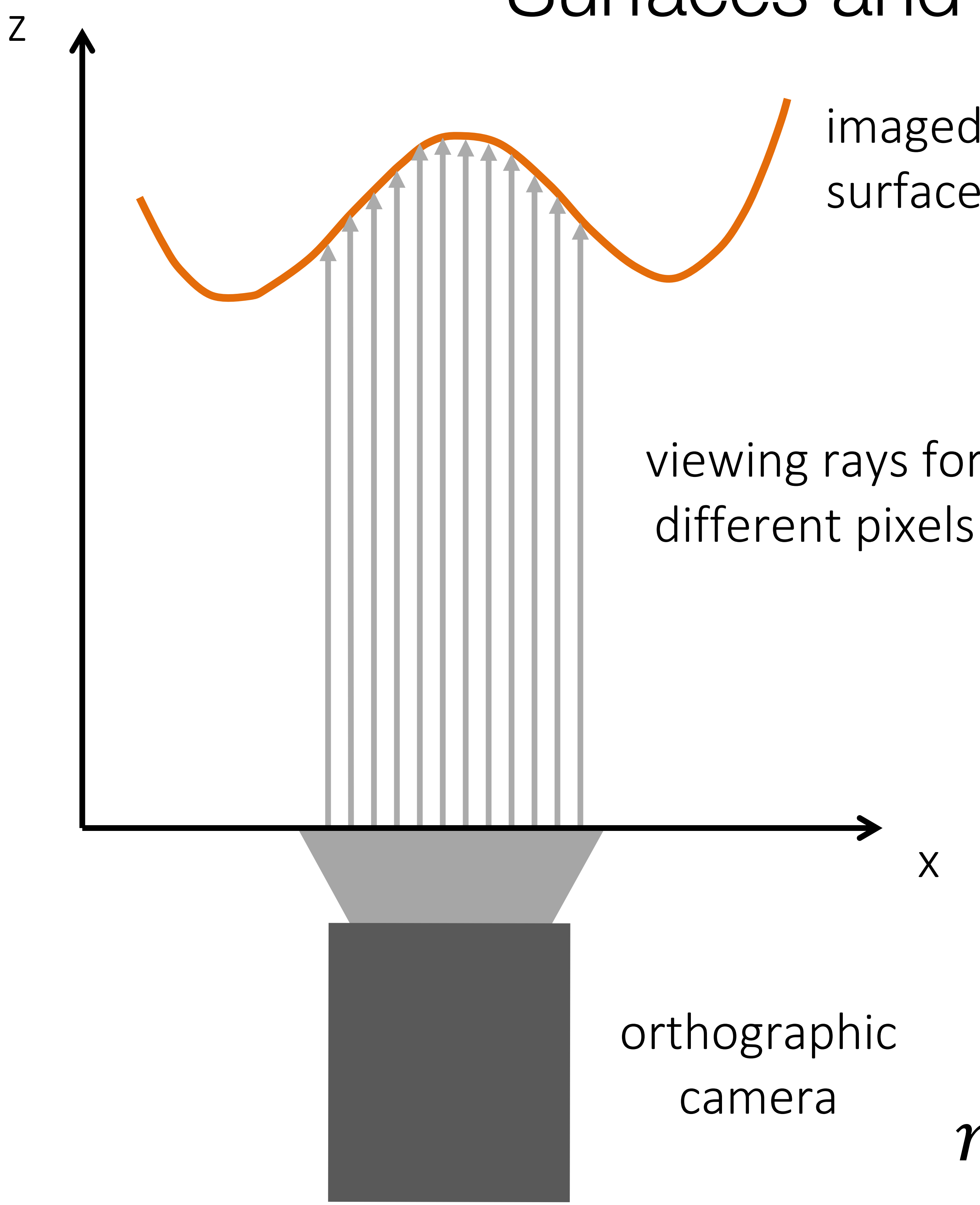


Estimated normals

Estimated normals (needle diagram)

- How can we compute depth from normals?
    - Normals are like the “derivative” of the true depth

# Surfaces and normals




Surface representation as a depth image (also known as Monge surface):

$$\underset{\text{depth at each pixel}}{z} = f\underbrace{(x, y)}_{\text{pixel coordinates in image space}}$$

Unnormalized normal:

$$\tilde{n}(x, y) = \left(\frac{df}{dx}, \frac{df}{dy}, -1\right)$$

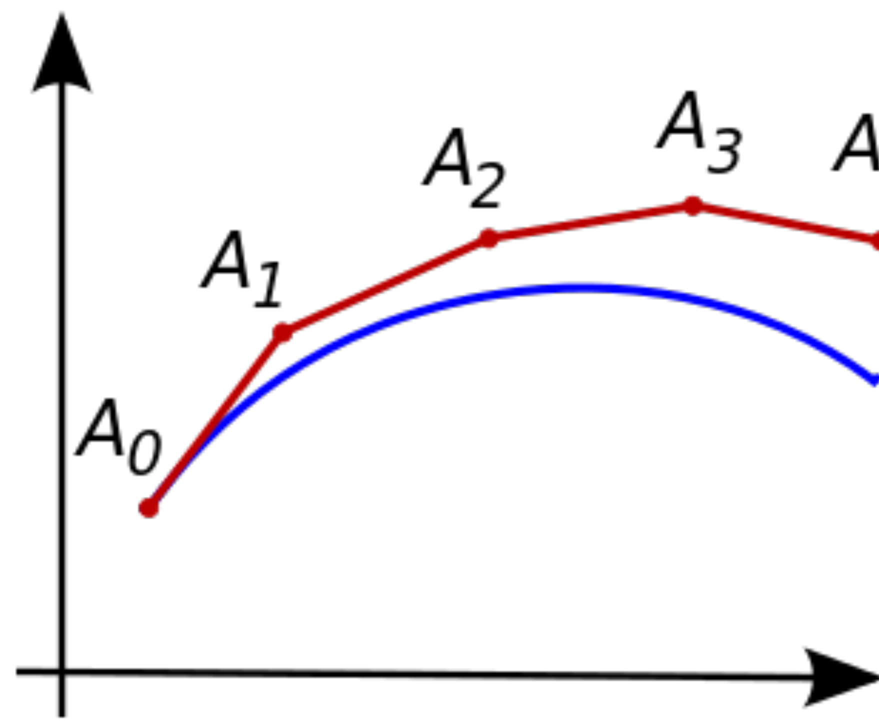
Actual normal:

$$n(x, y) = \tilde{n}(x, y) / \|\tilde{n}(x, y)\|$$

Normals are scaled spatial derivatives of depth image!

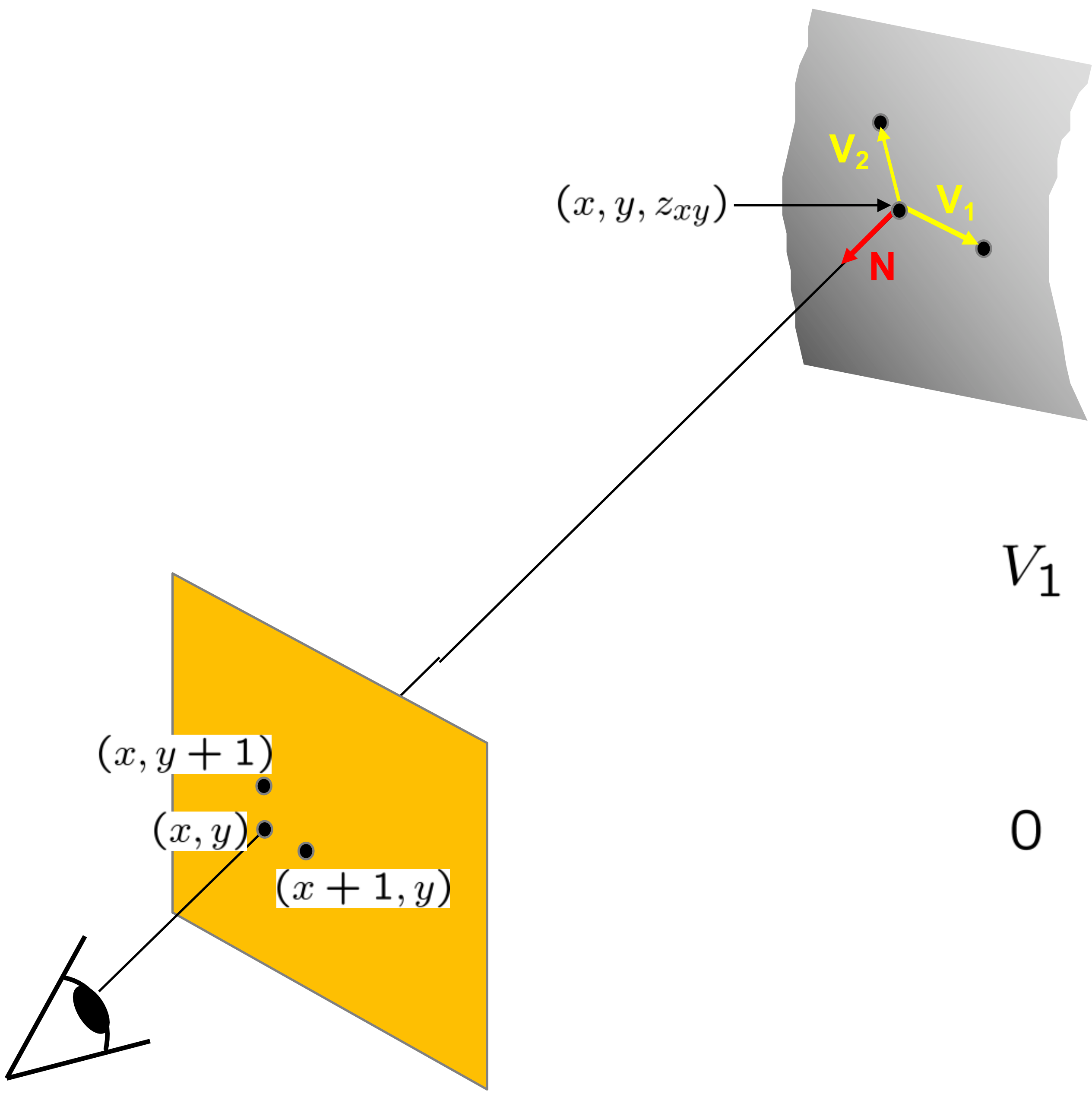
# Normal Integration

- Integrating a set of derivatives is easy in 1D
  - (similar to Euler's method from diff. eq. class)




- Could just integrate normals in each column / row separately
- Instead, we formulate as a linear system and solve for depths that *best agree with the surface normals*

# Depth from normals




$$
\begin{aligned}
V_1 &= (x+1, y, z_{x+1,y}) - (x, y, z_{xy}) \\
&= (1, 0, z_{x+1,y} - z_{xy})
\end{aligned}
$$

$$
\begin{aligned}
0 &= N \cdot V_1 \\
&= (n_x, n_y, n_z) \cdot (1, 0, z_{x+1,y} - z_{xy}) \\
&= n_x + n_z(z_{x+1,y} - z_{xy})
\end{aligned}
$$

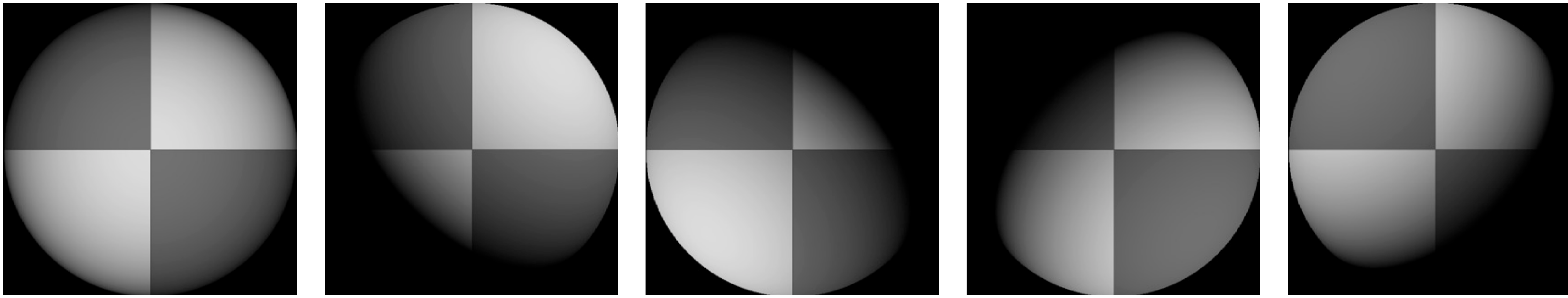
Get a similar equation for $\mathbf{V_2}$

- Each normal gives us two linear constraints on z
- compute z values by solving a matrix equation

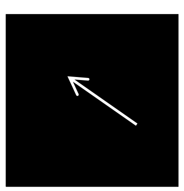
# Results

1. Estimate light source directions
2. Compute surface normals
3. Compute albedo values
4. Estimate depth from surface normals
5. Relight the object (with original texture and uniform albedo)

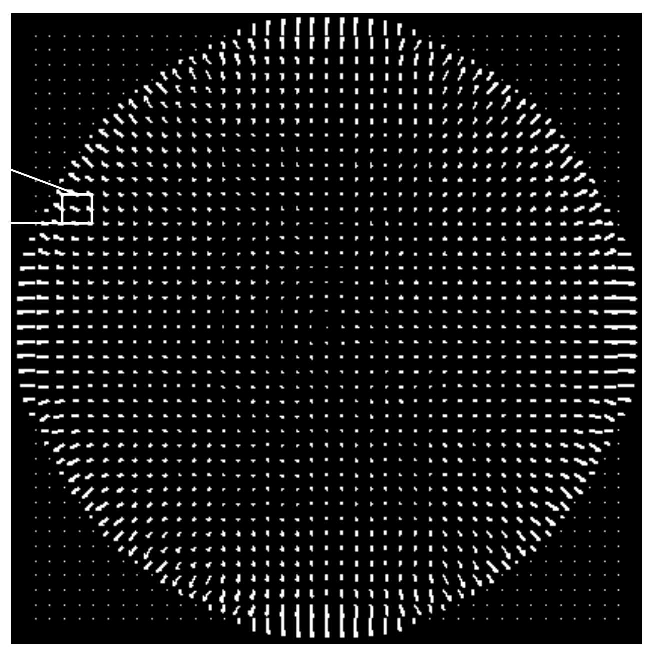
# Results: Lambertian Sphere



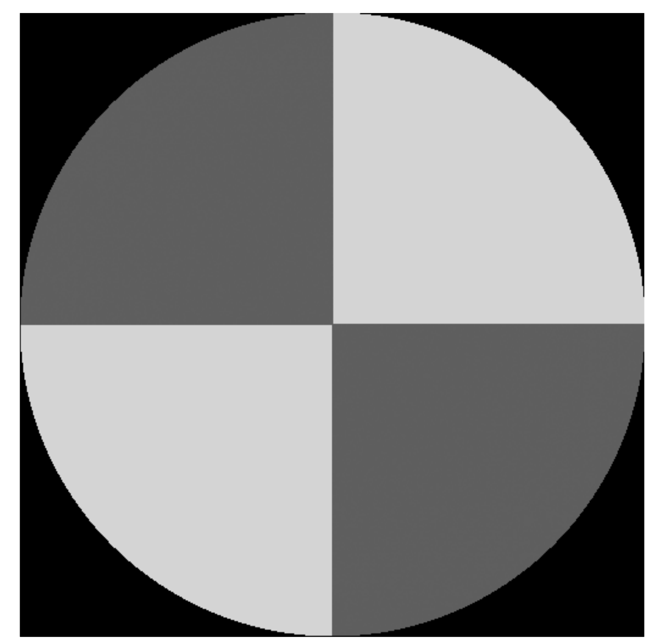
Input Images



Needles are projections of surface normals on image plane

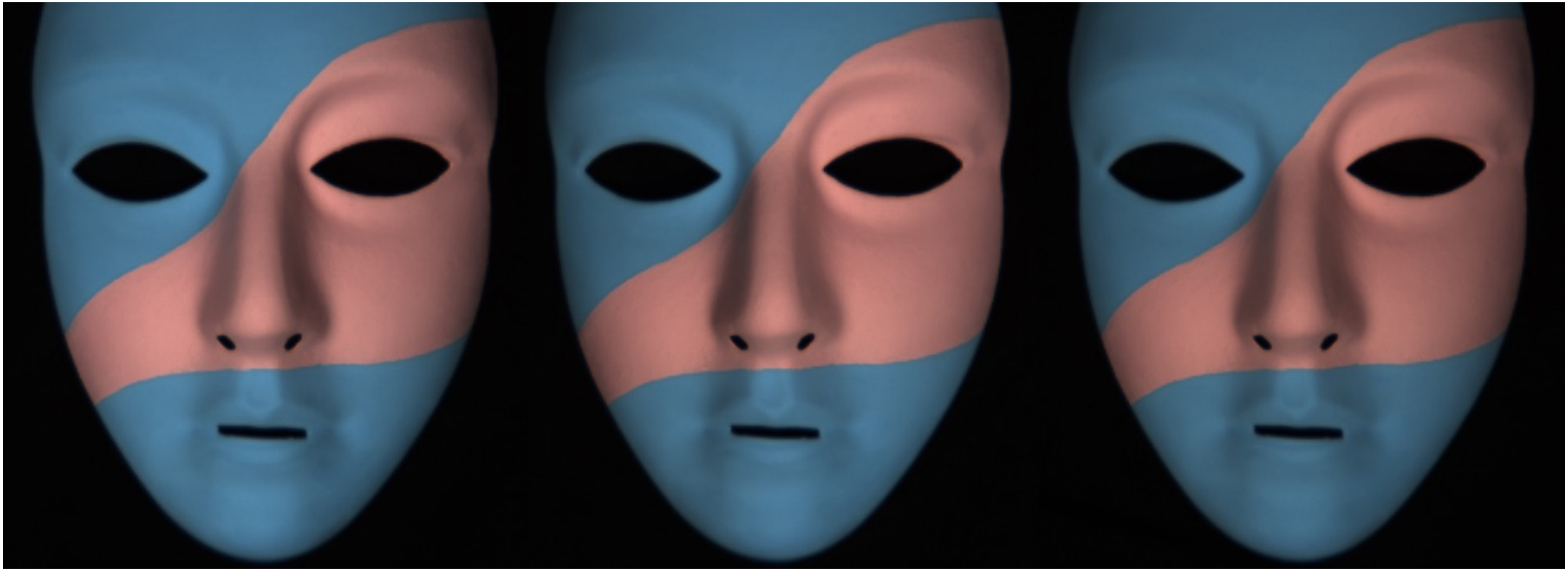


Estimated Surface Normals

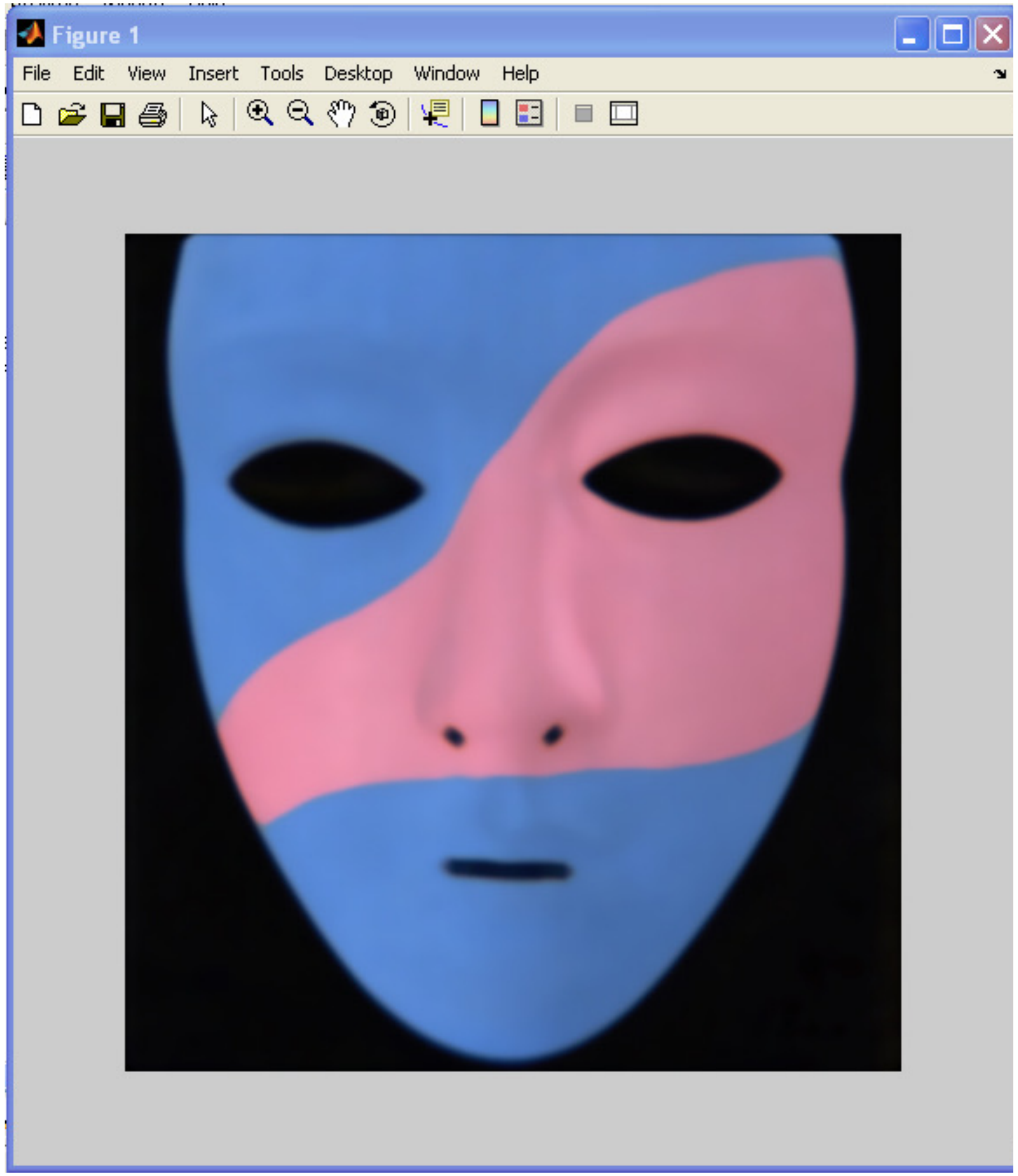


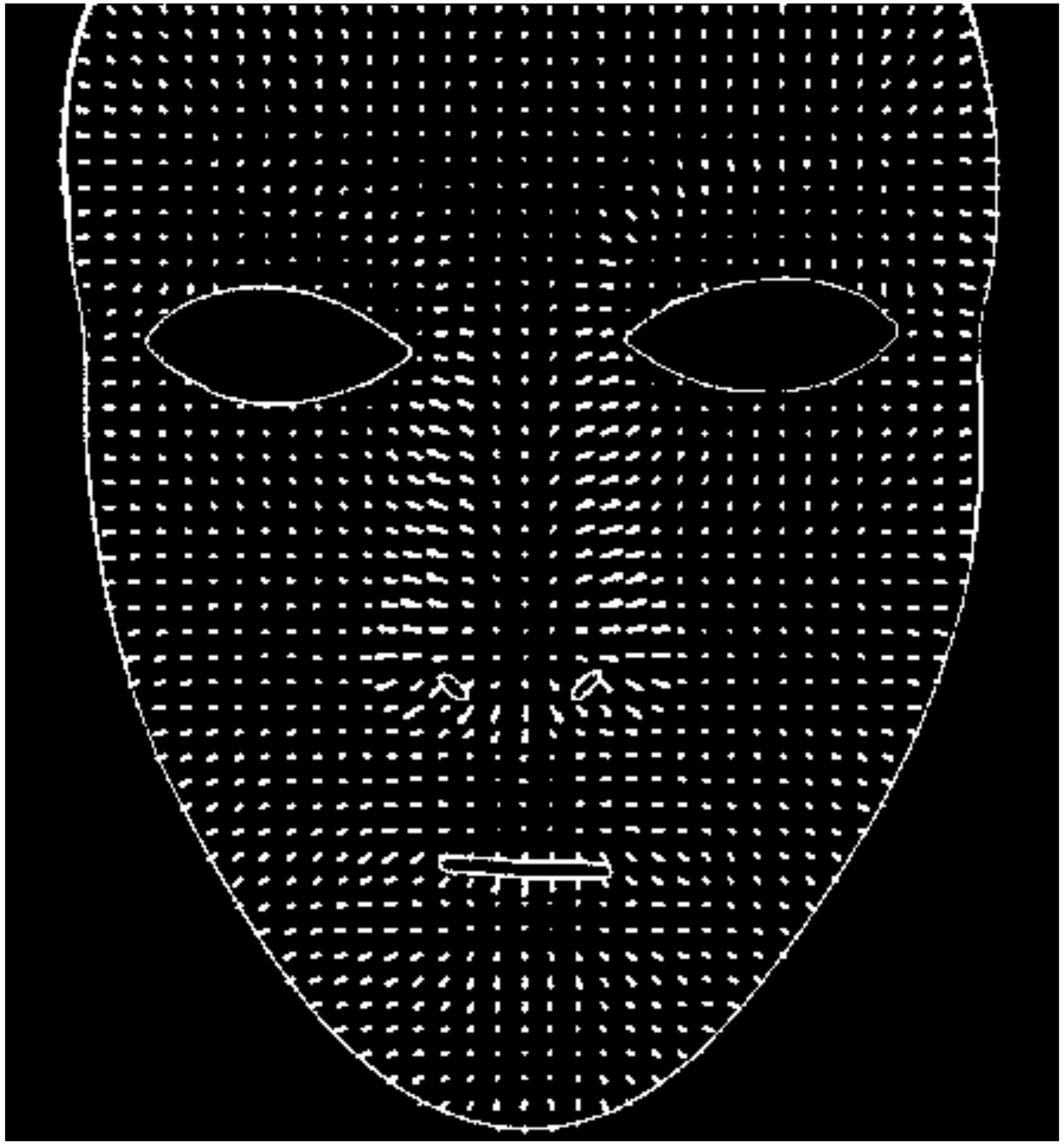
Estimated Albedo

# Lambertain Mask

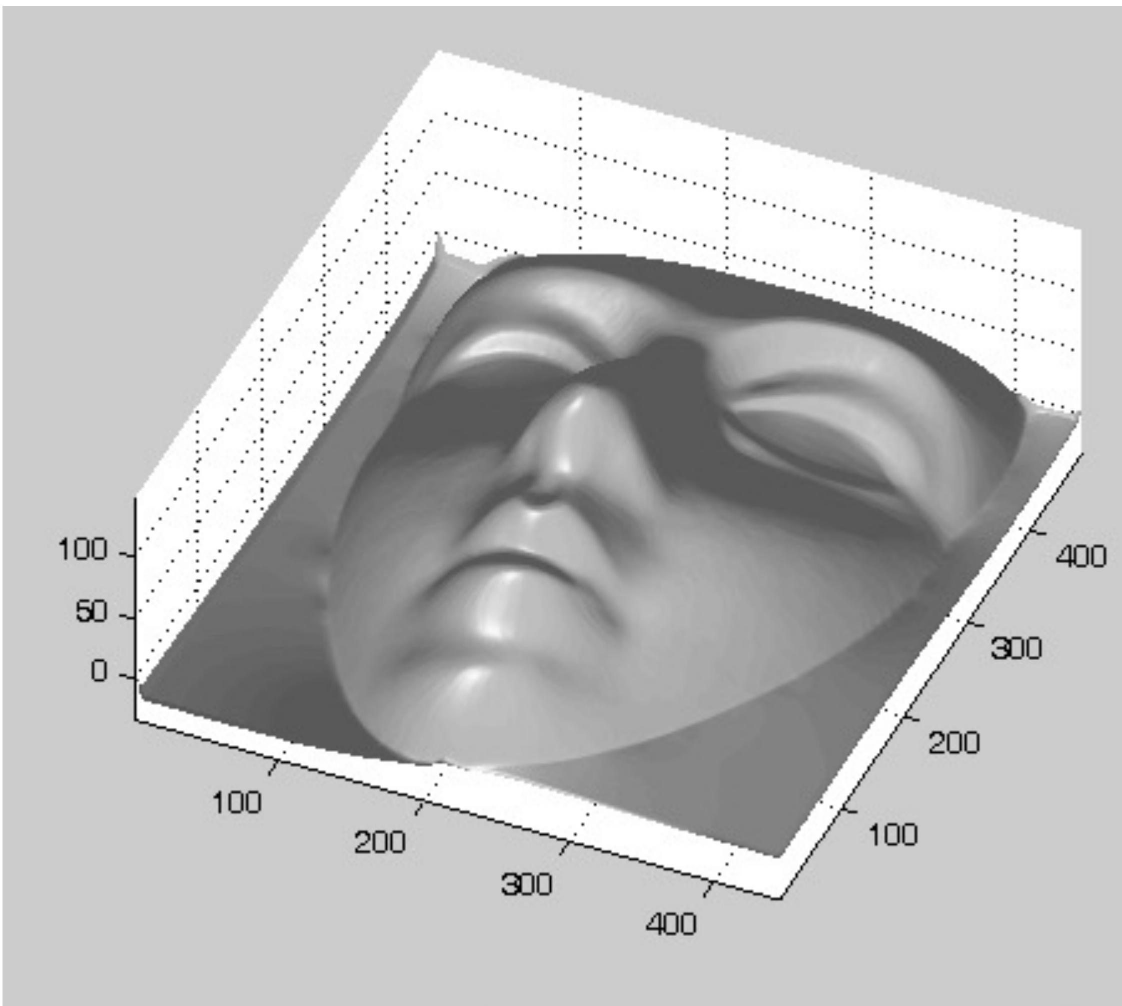


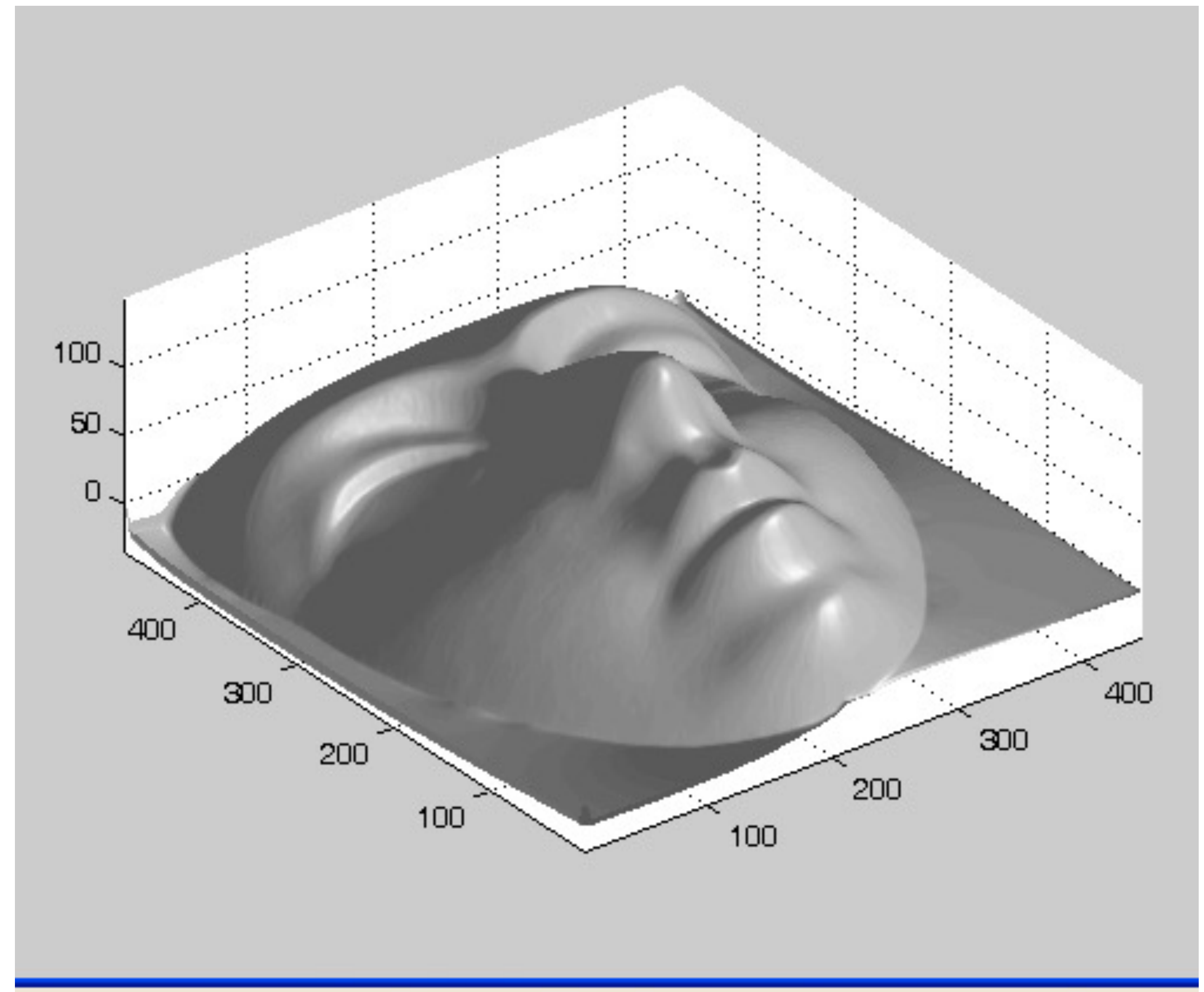
# Results – Albedo and Surface Normal



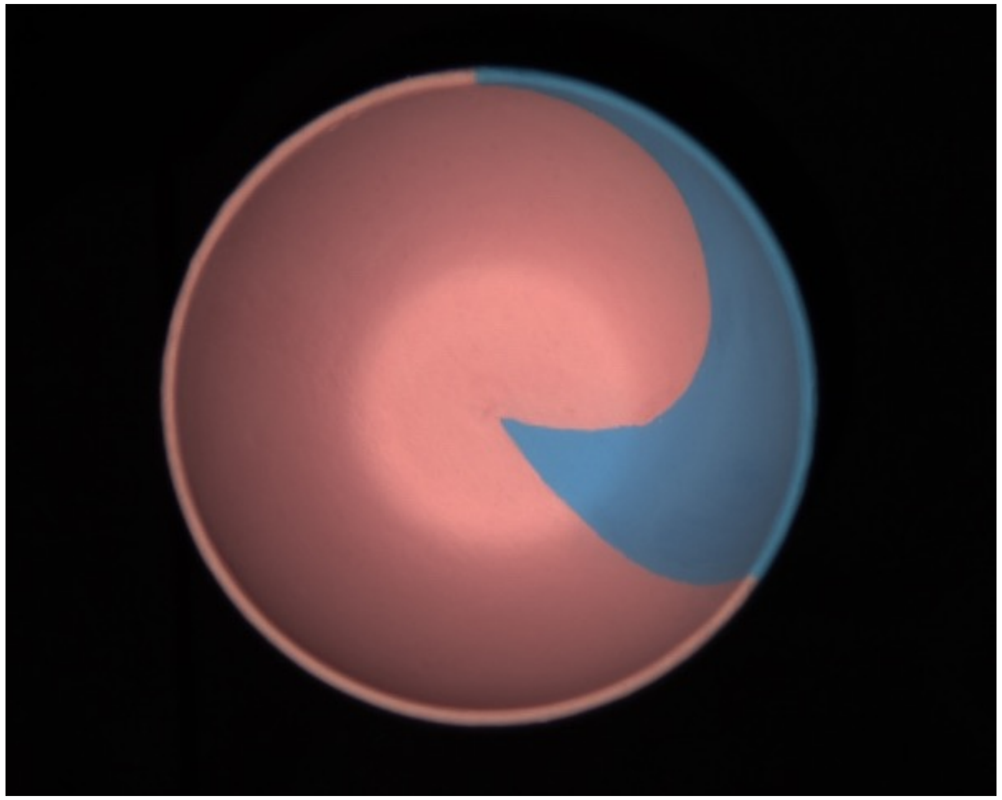
# Results – Shape of Mask

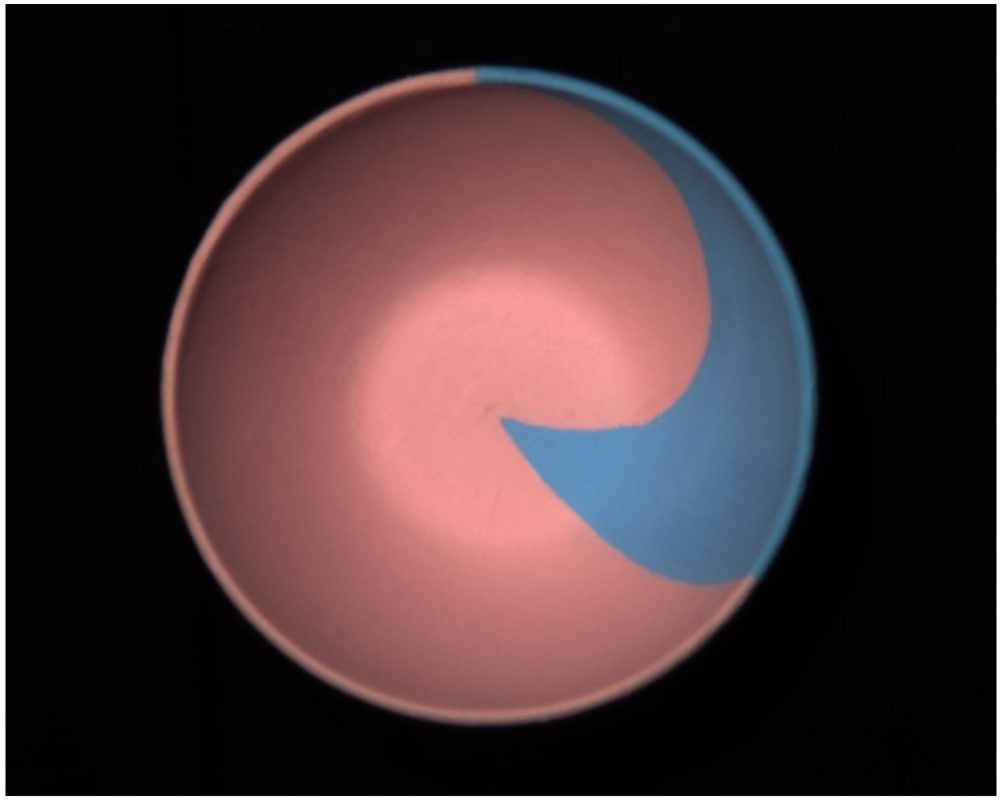


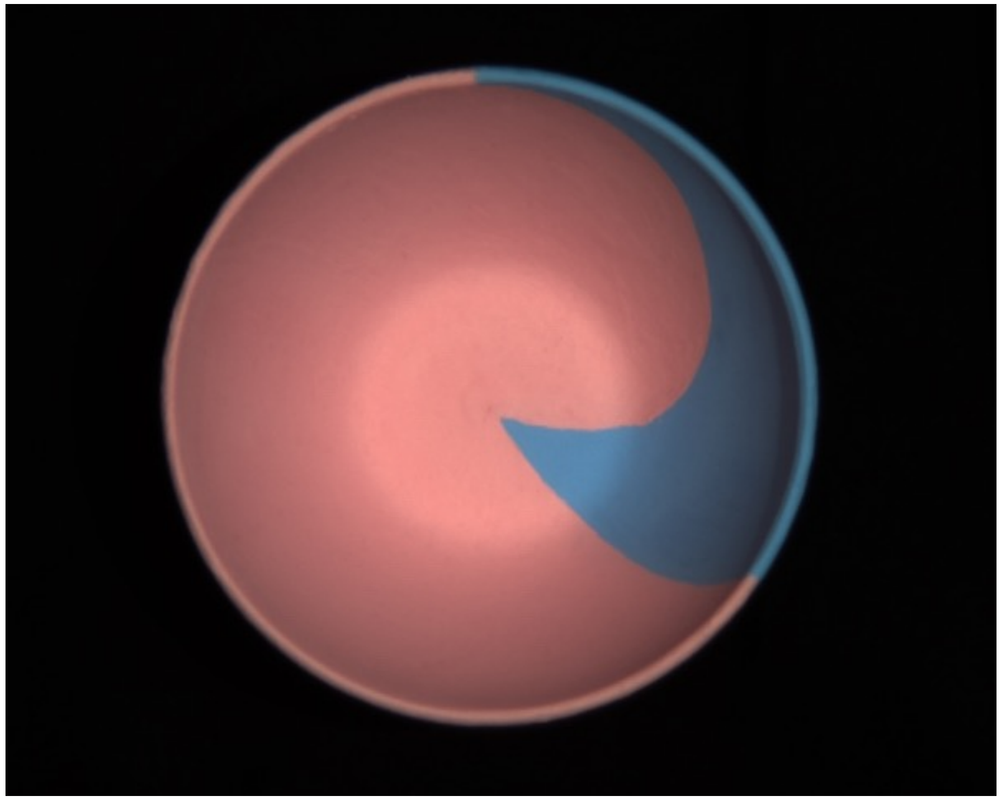



# Results: Lambertian Toy


# Non-idealities: interreflections







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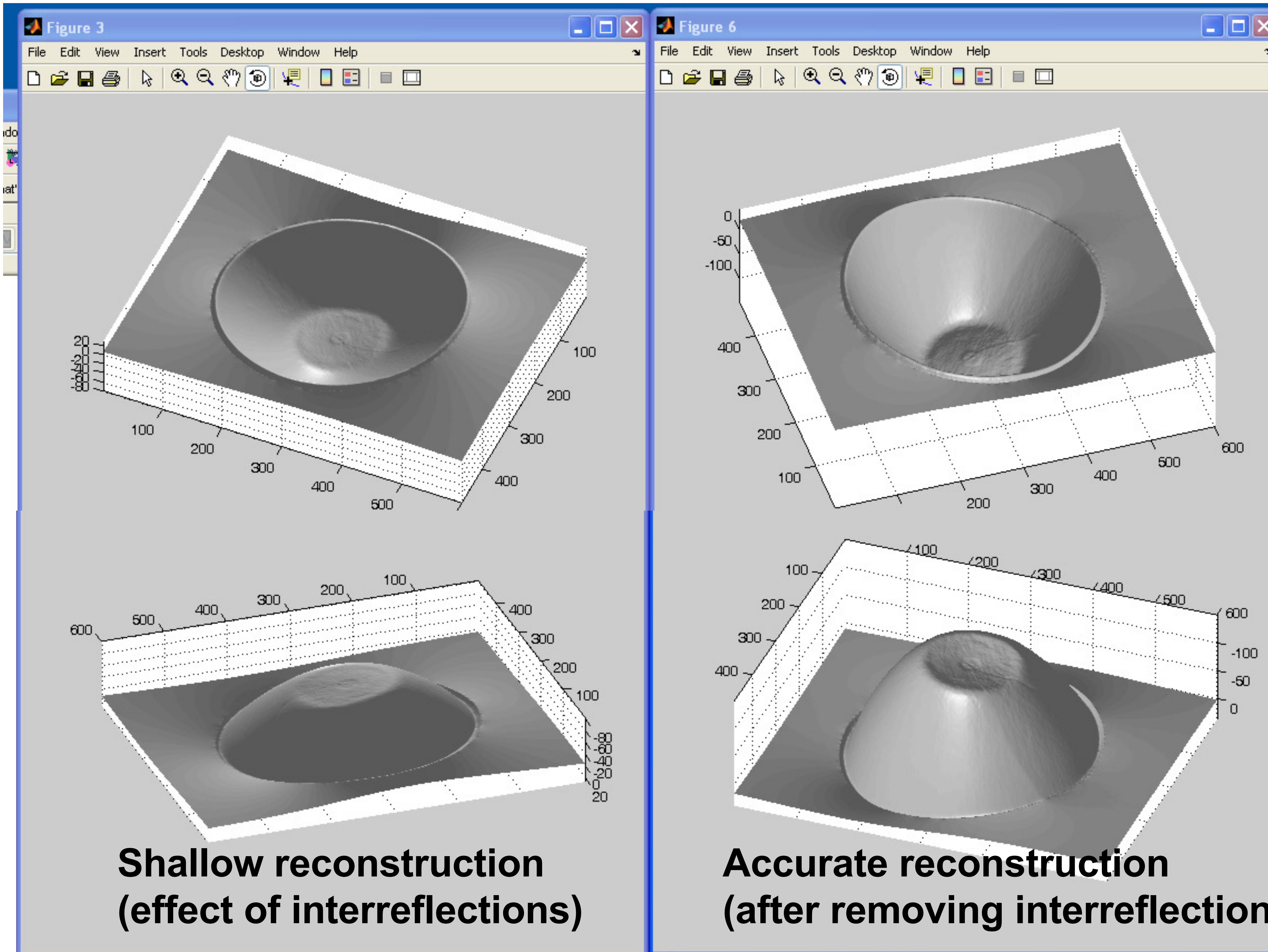



**Shallow reconstruction (effect of interreflections)**

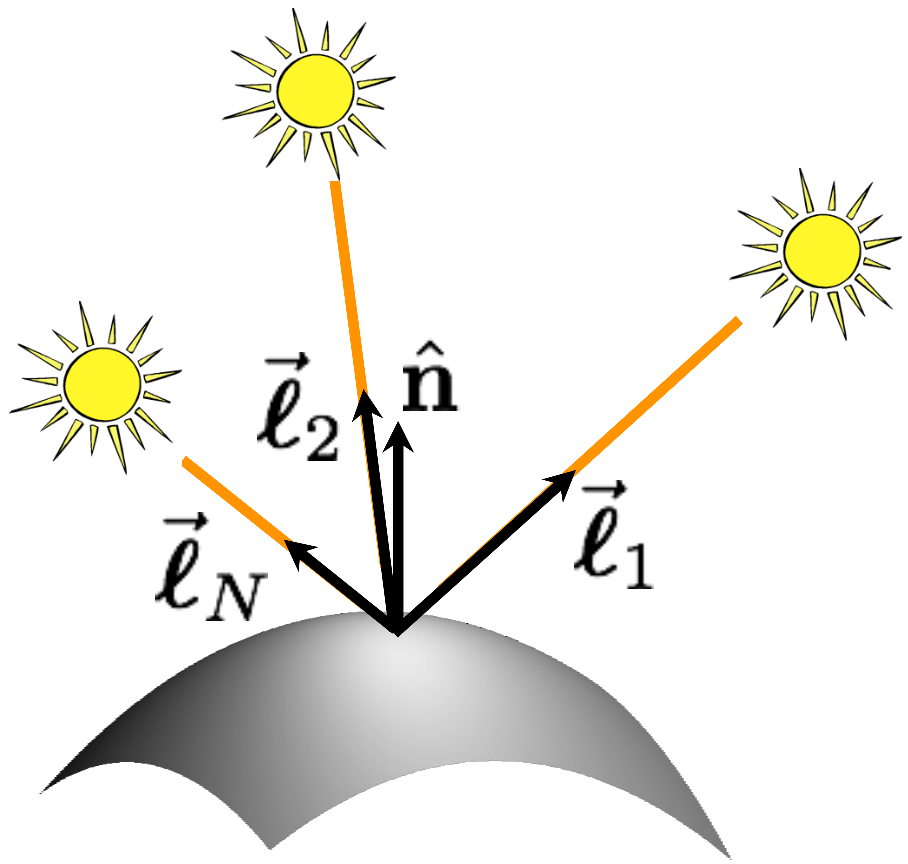
**Accurate reconstruction (after removing interreflections)**

# What if the light directions are unknown?

# Uncalibrated photometric stereo

# What if the light directions are unknown?

$$I_1 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_1$$
$$I_2 = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_2$$
$$\vdots$$
$$I_N = a\hat{\boldsymbol{n}}^\top \vec{\boldsymbol{\ell}}_N$$




define "pseudo-normal" $\vec{\boldsymbol{b}} \triangleq a\hat{\boldsymbol{n}}$

solve linear system for pseudo-normal

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} \vec{\boldsymbol{\ell}}_1^\top \\ \vec{\boldsymbol{\ell}}_2^\top \\ \vdots \\ \vec{\boldsymbol{\ell}}_N^\top \end{bmatrix}_{N\times 3} \begin{bmatrix} \vec{\boldsymbol{b}} \end{bmatrix}_{3\times 1}$$

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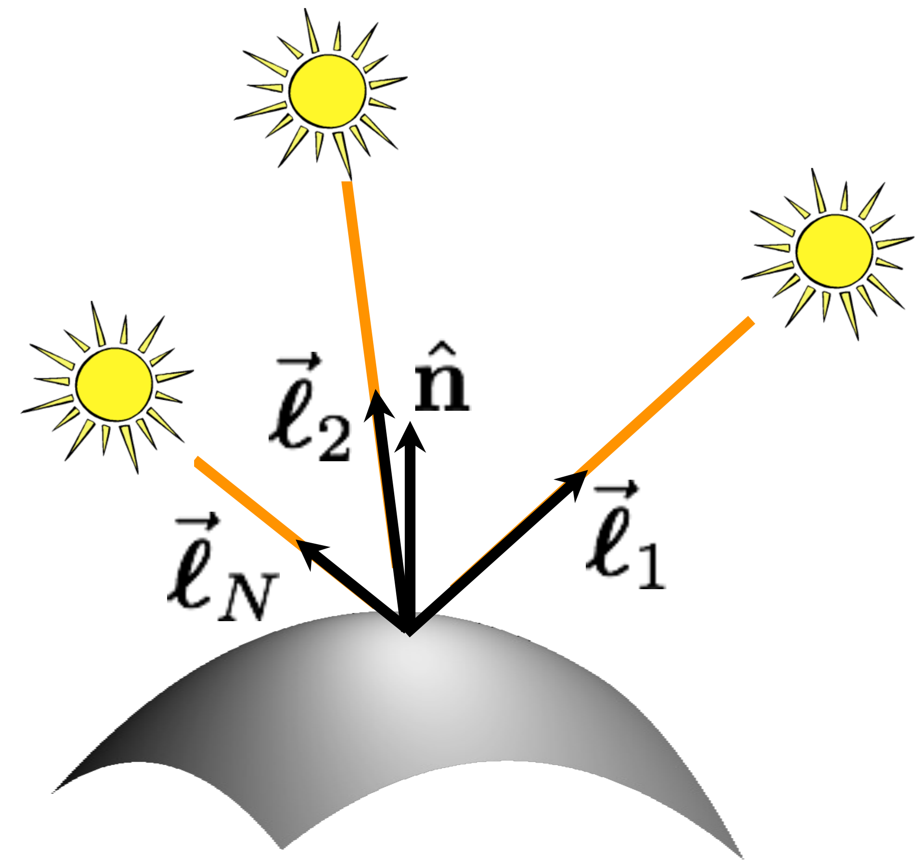


define “pseudo-normal” $\vec{\boldsymbol{b}} \triangleq a\hat{\boldsymbol{n}}$

solve linear system for pseudo-normal at each image pixel

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}_{N\times M} = \begin{bmatrix} \vec{\boldsymbol{\ell}}_1^\top \\ \vec{\boldsymbol{\ell}}_2^\top \\ \vdots \\ \vec{\boldsymbol{\ell}}_N^\top \end{bmatrix}_{N\times 3} \begin{bmatrix} B \end{bmatrix}_{3\times M}$$

M: number of pixels

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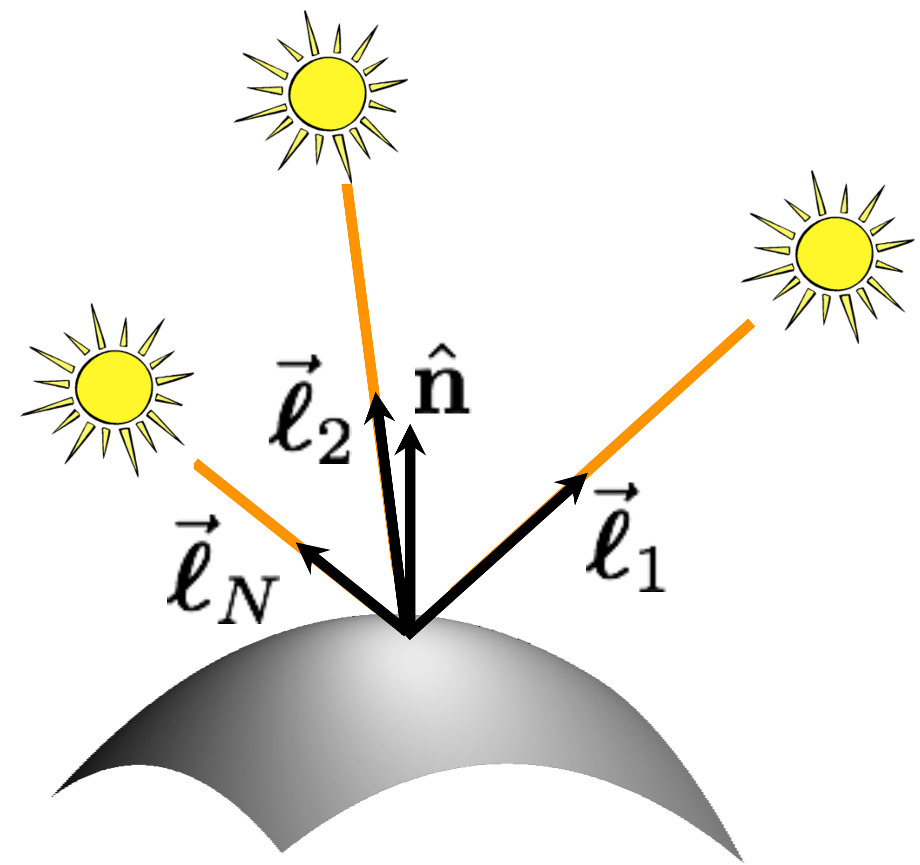


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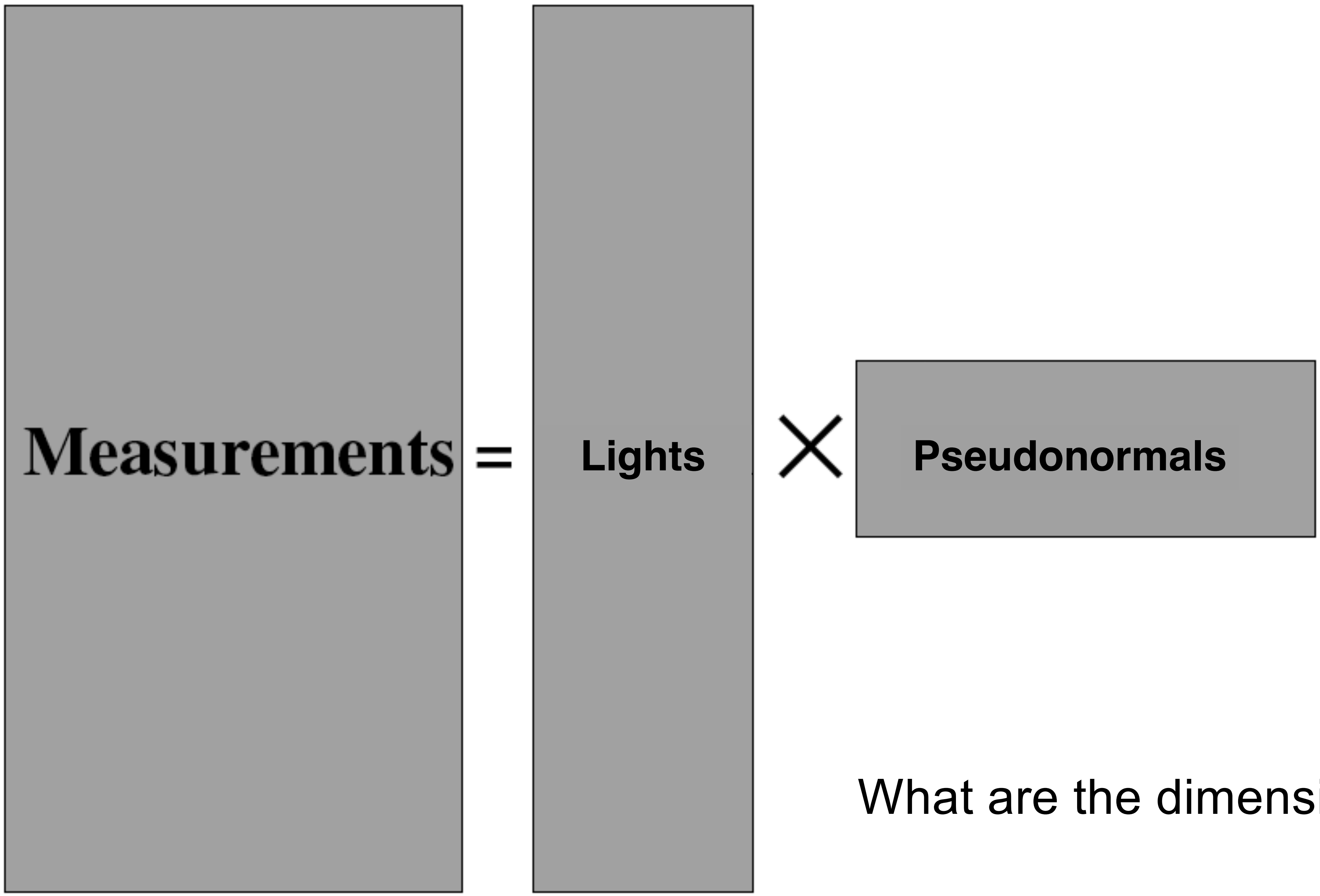
How do we solve this system without knowing light matrix L?

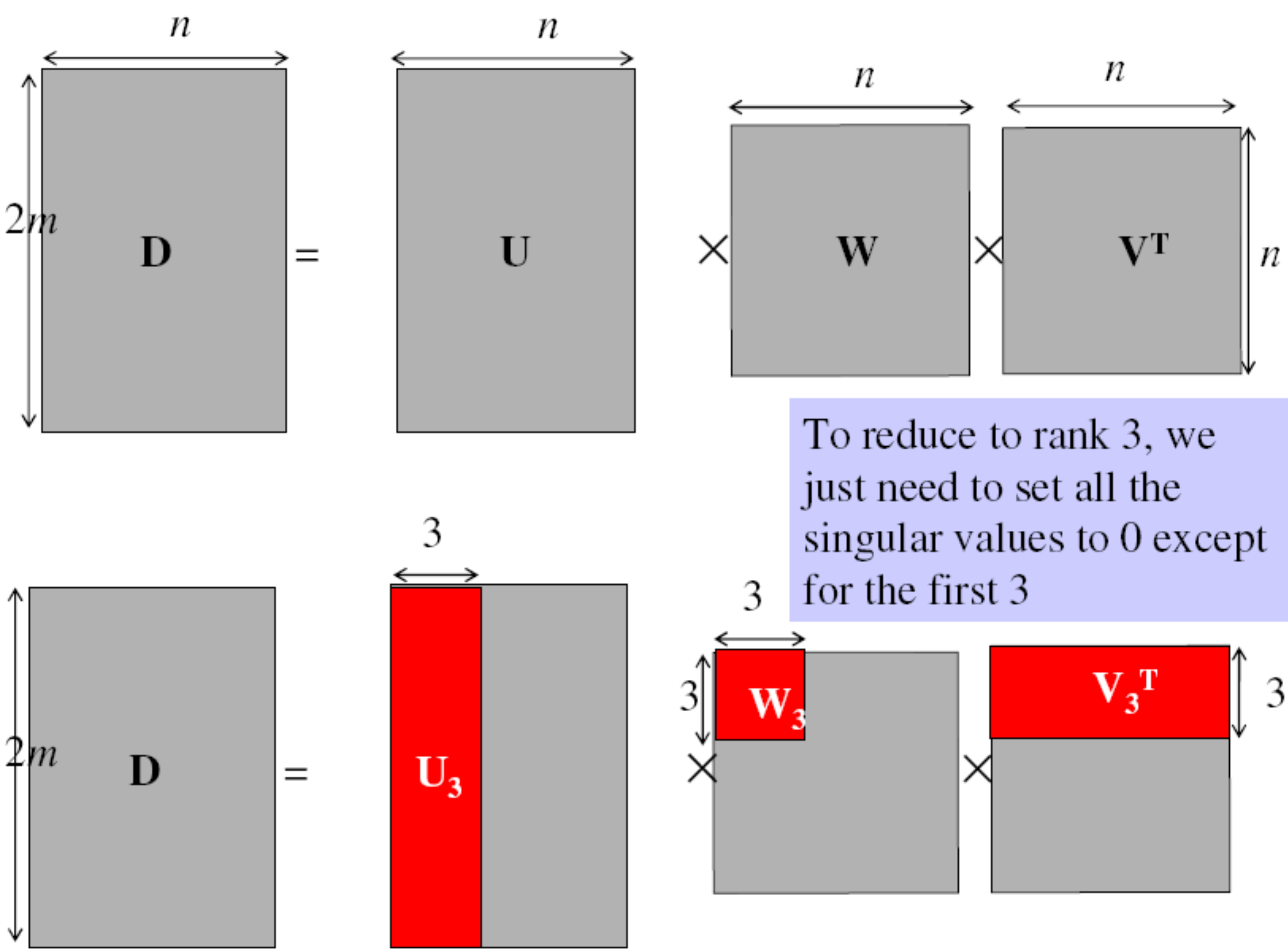
# Factorizing the measurement matrix




What are the dimensions?

# Factorizing the measurement matrix

- Singular value decomposition:




To reduce to rank 3, we just need to set all the singular values to 0 except for the first 3

This decomposition minimizes $|\mathbf{I}\text{-}\mathbf{LB}|^2$

# Are the results unique?

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We can insert any 3x3 matrix Q in the decomposition and get the same images:

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Can we use any assumptions to remove some of these 9 degrees of freedom?

# Generalized bas-relief ambiguity

# Enforcing integrability

What does the matrix **B** correspond to?

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What does the matrix **B** correspond to?

- Surface representation as a depth image (also known as Monge surface):

$$\underset{\uparrow}{z} = f(\underbrace{x, y})$$

depth at each pixel        pixel coordinates in image space

- Unnormalized normal:

$$\tilde{n}(x, y) = \left(\frac{df}{dx}, \frac{df}{dy}, -1\right)$$

- Actual normal:

$$n(x, y) = \tilde{n}(x, y) / \|\tilde{n}(x, y)\|$$

- Pseudo-normal:

$$b(x, y) = a(x, y) n(x, y)$$

- Rearrange into 3xN matrix **B**.

# Enforcing integrability

What does the integrability constraint correspond to?

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- Differentiation order should not matter:

$$\frac{d}{dy}\frac{df(x,y)}{dx} = \frac{d}{dx}\frac{df(x,y)}{dy}$$

- Can you think of a way to express the above using pseudo-normals **b**?

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$$\frac{d}{dy}\frac{b_1(x,y)}{b_3(x,y)} = \frac{d}{dx}\frac{b_2(x,y)}{b_3(x,y)}$$

- Simplify to:

$$b_3(x,y)\frac{db_1(x,y)}{dy} - b_1(x,y)\frac{db_3(x,y)}{dy} = b_2(x,y)\frac{db_1(x,y)}{dx} - b_1(x,y)\frac{db_2(x,y)}{dx}$$

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- If $\mathbf{B_e}$ is the pseudo-normal matrix we get from SVD, then find the 3x3 transform **D** such that $\mathbf{B=D \cdot B_e}$ is the closest to satisfying integrability in the least-squares sense.

# Enforcing integrability

Does enforcing integrability remove all ambiguities?

# Generalized Bas-relief ambiguity

If **B** is integrable, then:

- **B'=G$^{-T}$·B** is also integrable for all **G** of the form ($\lambda \neq 0$)

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & \nu & \lambda \end{bmatrix}$$

- Combined with transformed lights **S'=G·S**, the transformed pseudonormals produce the same images as the original pseudonormals.

- This ambiguity cannot be removed using shadows.

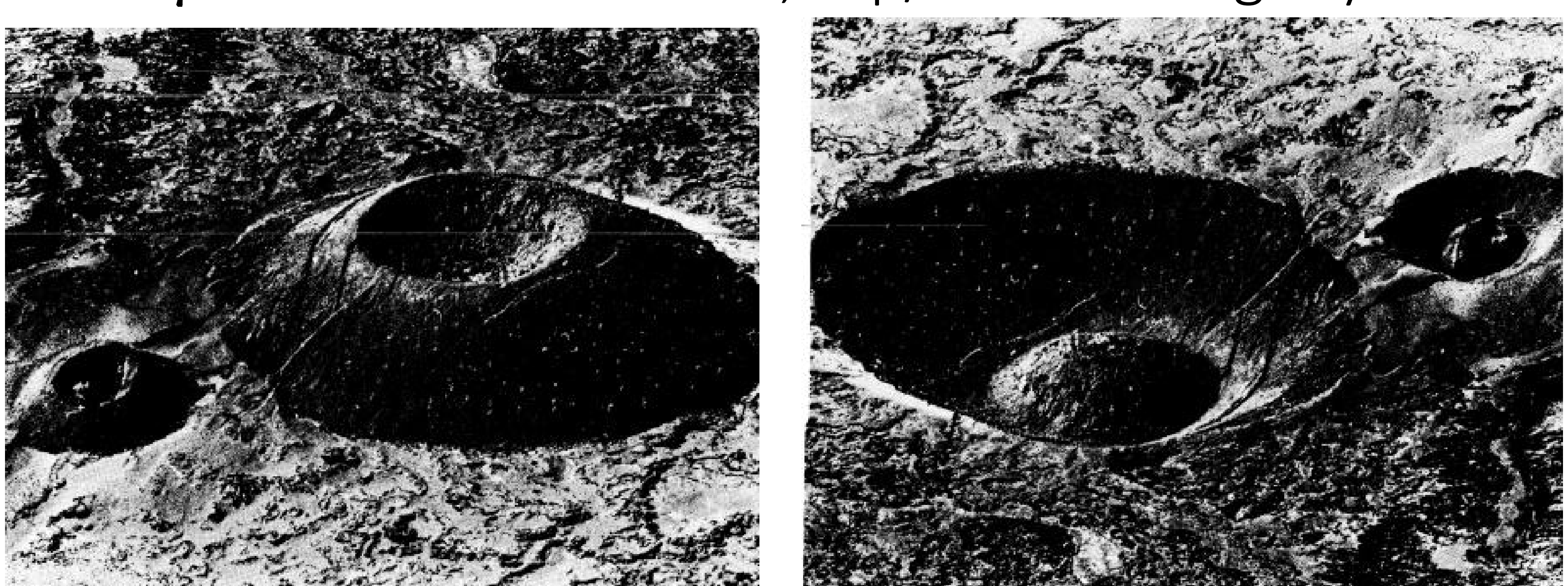
- This ambiguity *can* be removed using interreflections or additional assumptions.

This ambiguity is known as the generalized bas-relief ambiguity.

# Generalized Bas-relief ambiguity

When $\mu = \nu = 0$, **G** is equivalent to the transformation employed by relief sculptures.

When $\mu = \nu = 0$ and $\lambda = +\text{-}1$, top/down ambiguity.



Otherwise, includes shearing.

# What assumptions have we made for all this?

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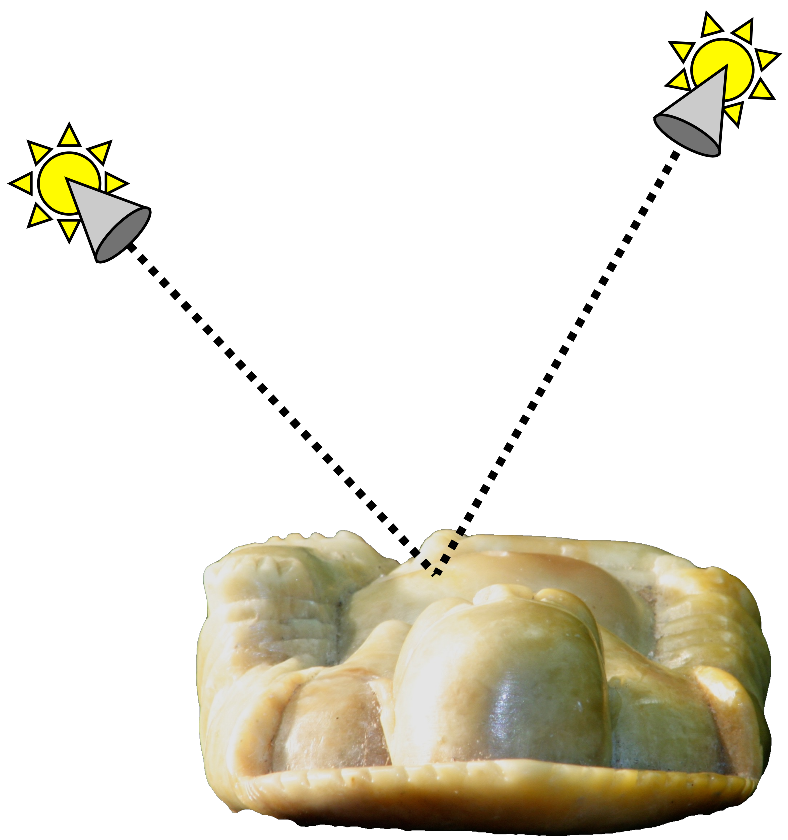
- Lambertian BRDF
- Directional lighting
- Orthographic camera
- No interreflections or scattering

# Shape independent of BRDF via reciprocity: "Helmholtz Stereopsis"



$$I = f(\text{shape}, \cancel{\text{illumination}}, \cancel{\text{reflectance}})$$

$$f^{-1} =$$

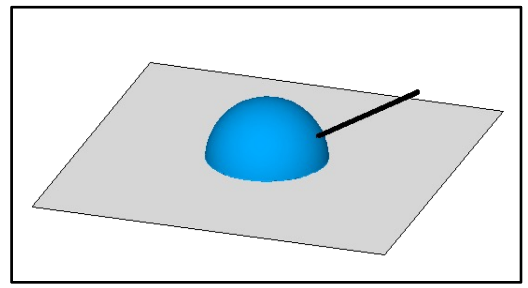
[Zickler et al., 2002]

# Shape from shading

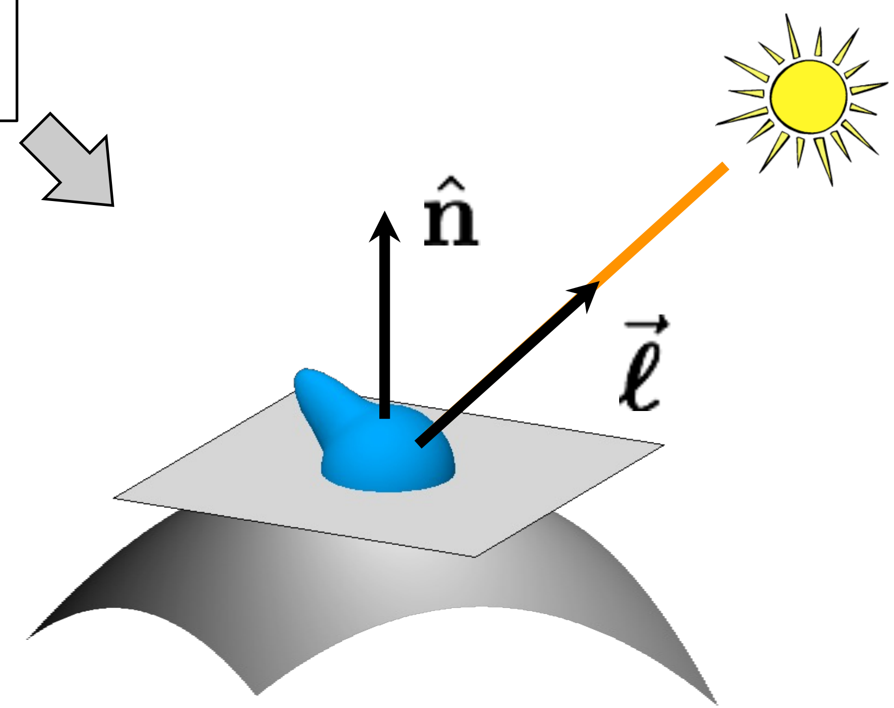
# Can we reconstruct shape from one image?

# Single-lighting is ambiguous

ASSUMPTION 1:
LAMBERTIAN

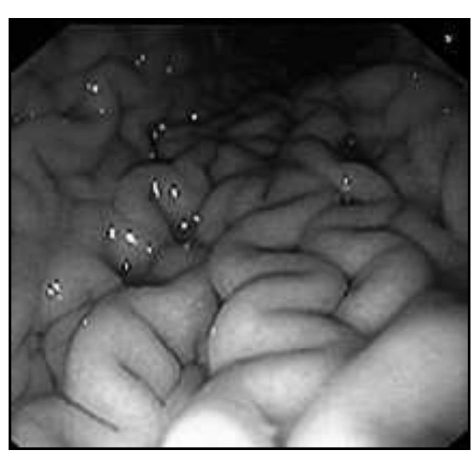


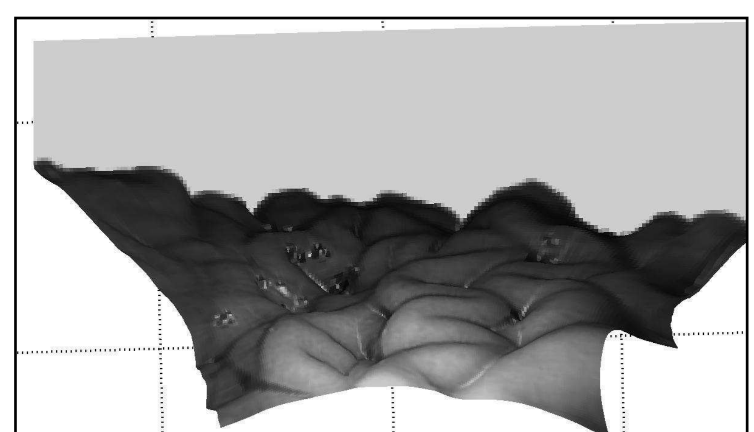
ASSUMPTION 2:
DIRECTIONAL LIGHTING



$$L^{\text{out}}(\hat{\boldsymbol{\omega}}) = \int_{\Omega_{\text{in}}} f(\hat{\boldsymbol{\omega}}_{\text{in}}, \hat{\boldsymbol{\omega}}_{\text{out}}) L^{\text{in}}(\hat{\boldsymbol{\omega}}_{\text{in}}) \cos\theta_{\text{in}} d\hat{\boldsymbol{\omega}}_{\text{in}}$$

$$\boxed{I = a\hat{\mathbf{n}}^{\top}\vec{\boldsymbol{\ell}}}$$

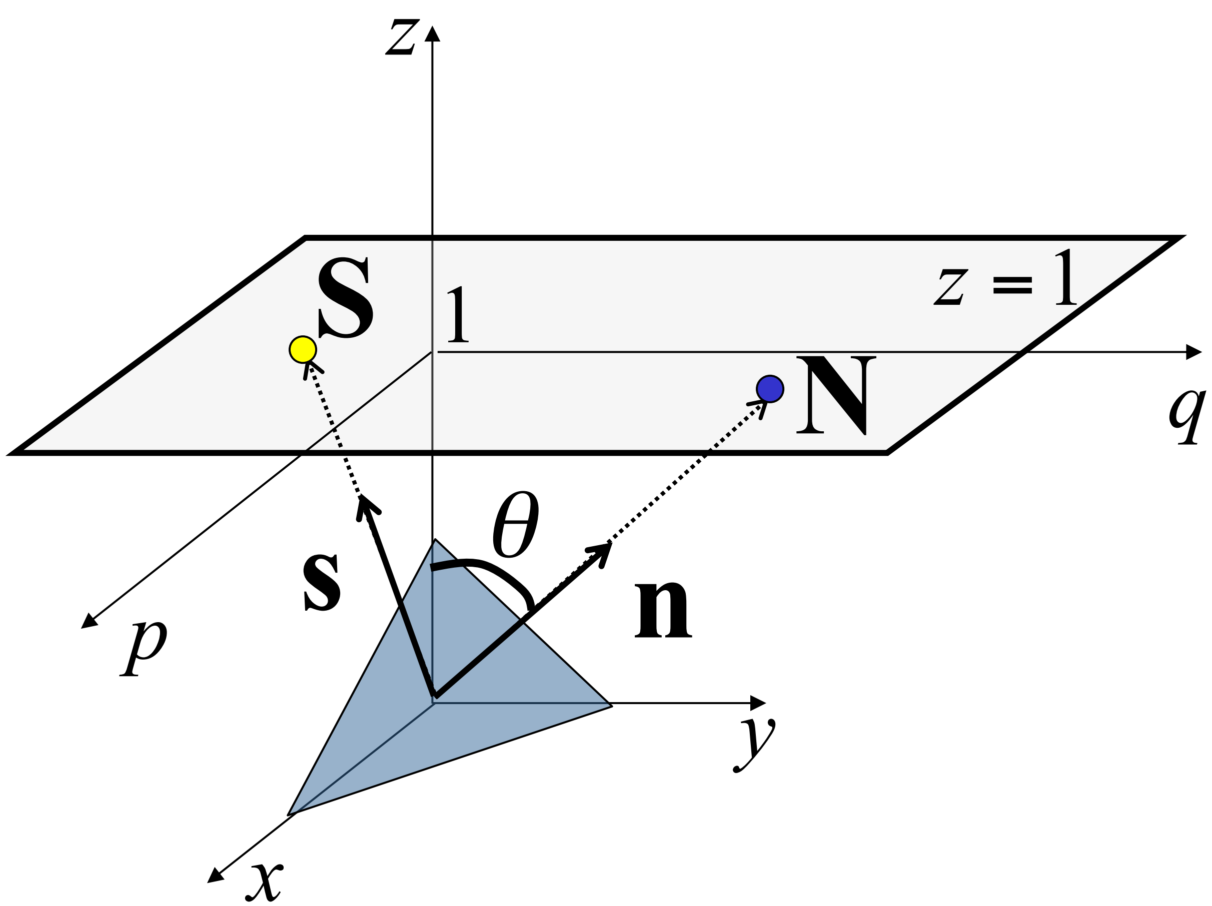




[Prados, 2004]

# Stereographic Projection

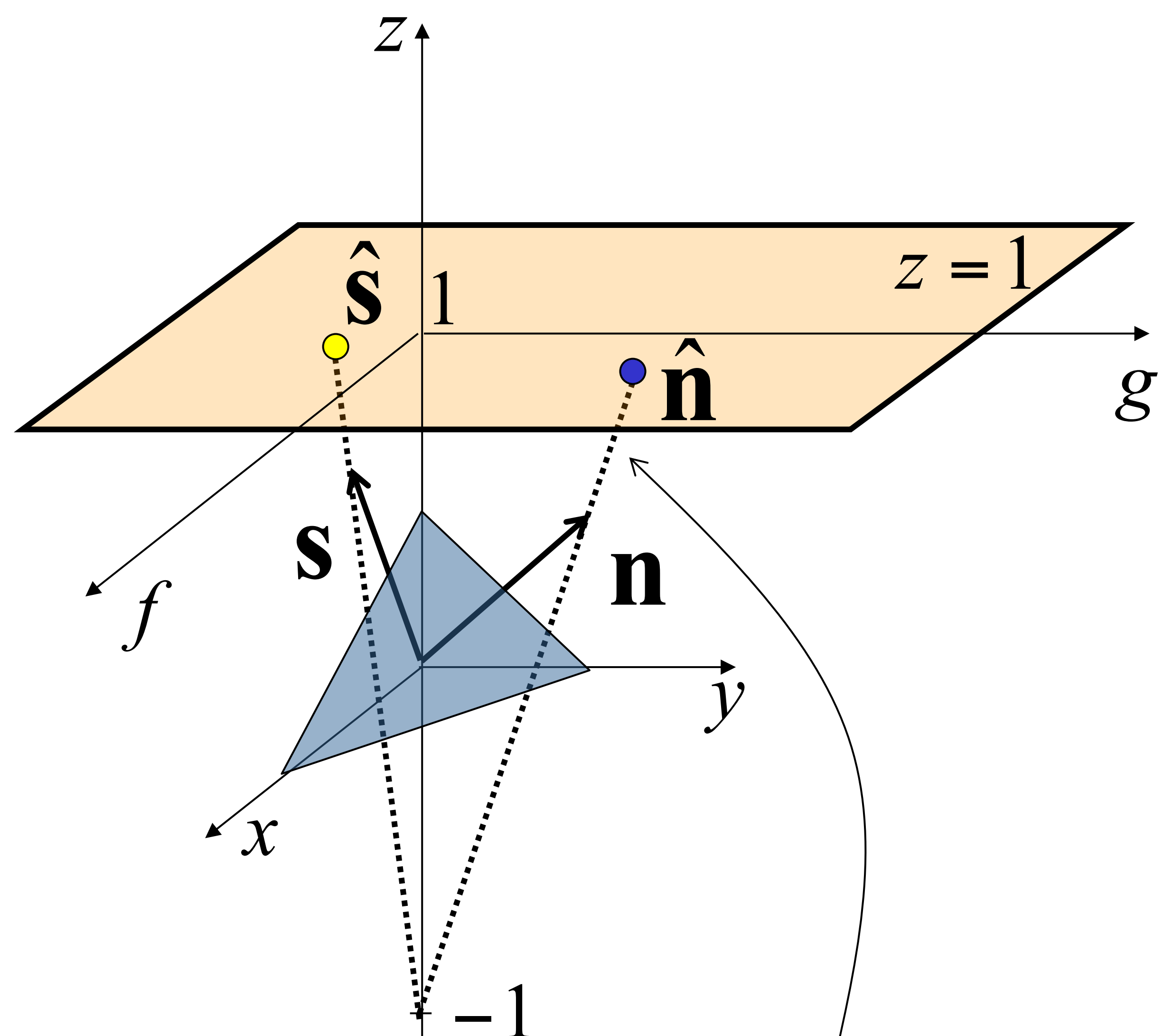
*(p,q)*-space (gradient space)




Problem

*(p,q)* can be infinite when $\theta = 90^\circ$

*(f,g)*-space




$$f = \frac{2p}{1+\sqrt{1+p^2+q^2}} \qquad g = \frac{2q}{1+\sqrt{1+p^2+q^2}}$$

Redefine reflectance map as $R(f,g)$

# Image Irradiance Constraint

- Image irradiance should match the reflectance map

Minimize

$$\boxed{e_i = \iint_{\text{image}} \left(I(x,y) - R(f,g)\right)^2 dx\,dy}$$

(minimize errors in image irradiance in the image)

# Smoothness Constraint

- Used to constrain shape-from-shading
- Relates orientations *(f,g)* of neighboring surface points

Minimize

$$\boxed{e_s = \iint_{\text{image}} \left(f_x^2 + f_y^2\right) + \left(g_x^2 + g_y^2\right) dxdy}$$

$(f, g)$: surface orientation under stereographic projection

$$f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, g_x = \frac{\partial g}{\partial x}, g_y = \frac{\partial g}{\partial y}$$

(penalize rapid changes in surface orientation *f* and *g* over the image)

# Shape-from-Shading

- Find surface orientations *(f,g)* at all image points that minimize

$$e = \underbrace{e_s}_{\text{smoothness constraint}} + \overbrace{\lambda}^{\text{weight}} \underbrace{e_i}_{\text{image irradiance error}}$$

Minimize

$$\boxed{e = \iint_{\text{image}} \left(f_x^2 + f_y^2\right) + \left(g_x^2 + g_y^2\right) + \lambda\left(I(x,y) - R(f,g)\right)^2 dx\,dy}$$

# Numerical Shape-from-Shading

- **Smoothness error** at image point *(i,j)*

$$s_{i,j} = \frac{1}{4}\left(\left(f_{i+1,j} - f_{i,j}\right)^2 + \left(f_{i,j+1} - f_{i,j}\right)^2 + \left(g_{i+1,j} - g_{i,j}\right)^2 + \left(g_{i,j+1} - g_{i,j}\right)^2\right)$$

Of course you can consider more neighbors (smoother results)

- **Image irradiance error** at image point *(i,j)*

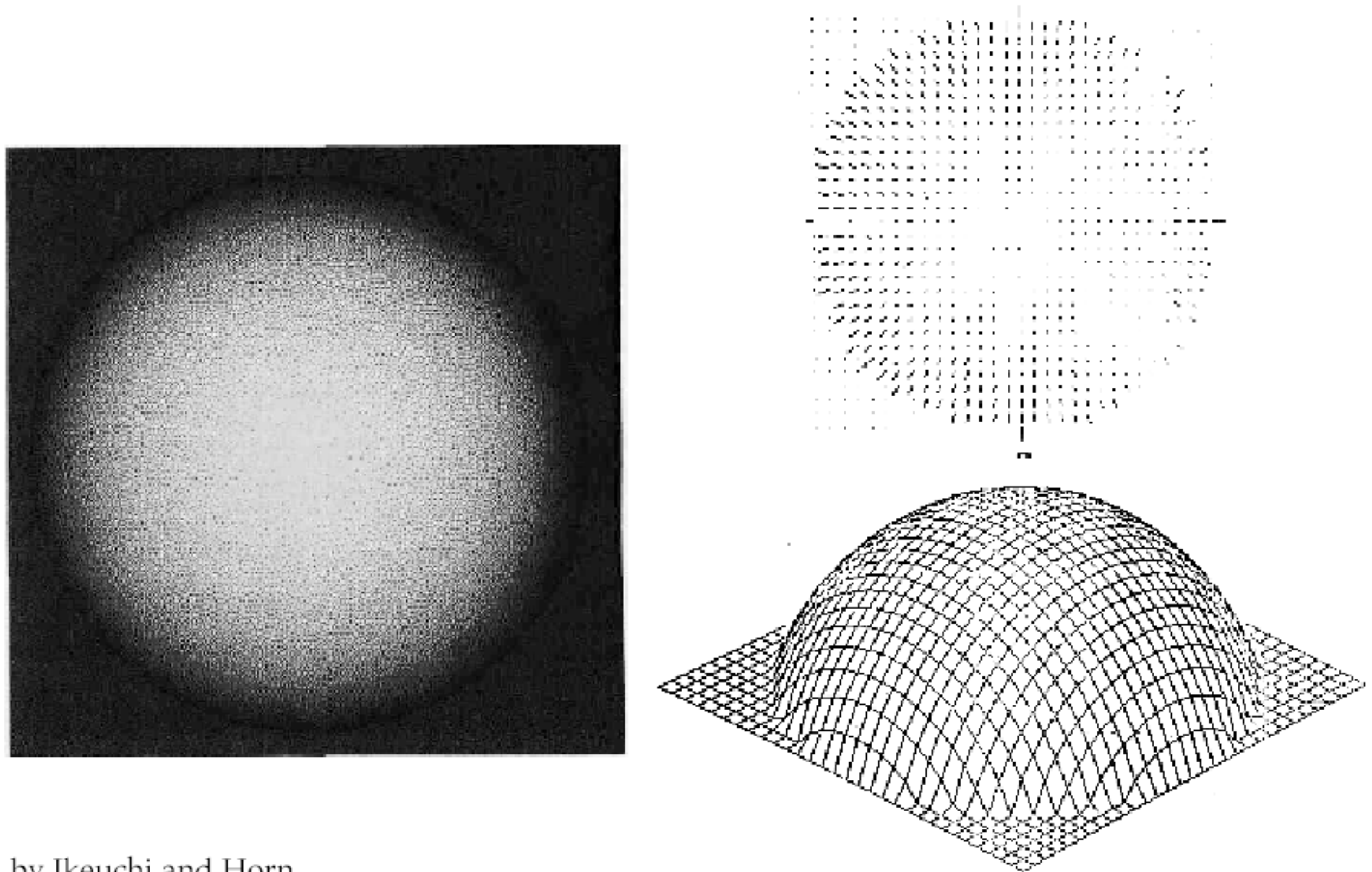
$$r_{i,j} = \left(I_{i,j} - R\left(f_{i,j}, g_{i,j}\right)\right)^2$$

Find $\left\{f_{i,j}\right\}$ and $\left\{g_{i,j}\right\}$ that minimize

$$e = \sum_i \sum_j \left(s_{i,j} + \lambda r_{i,j}\right)$$

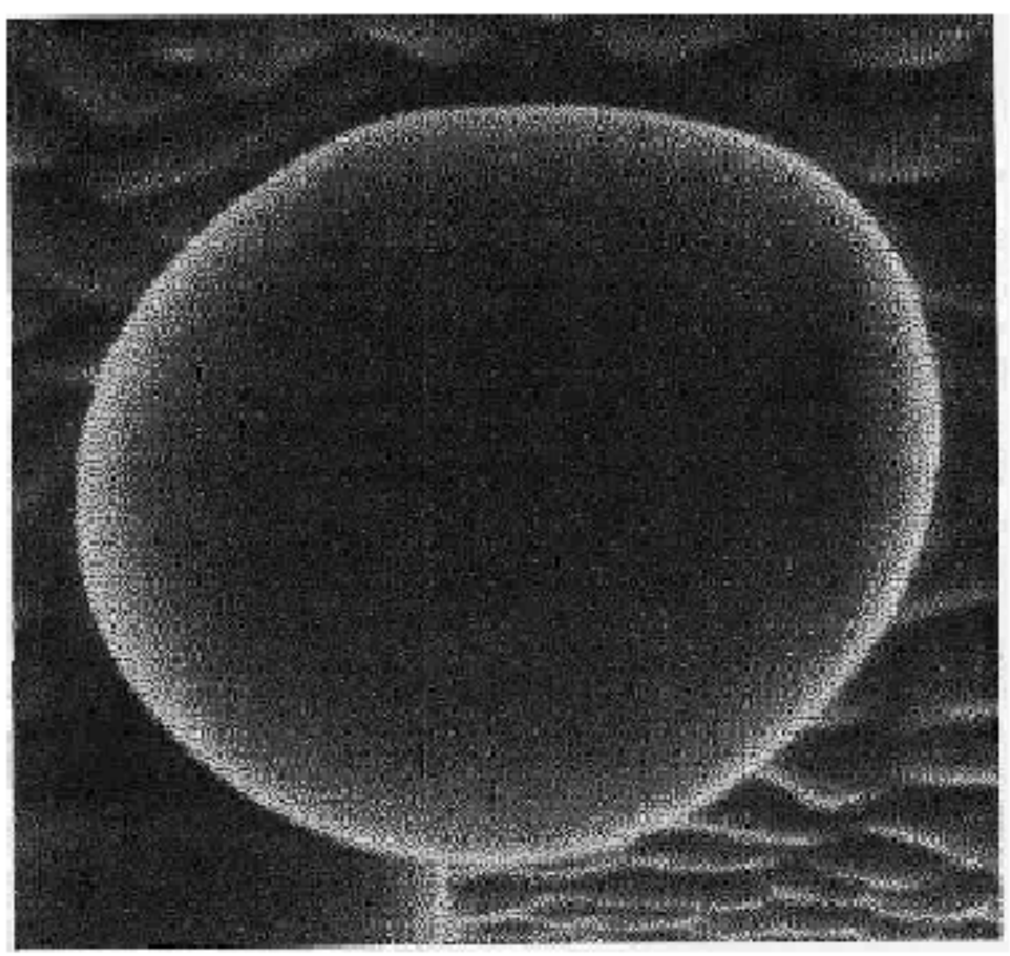
(Ikeuchi & Horn 89)

# Results

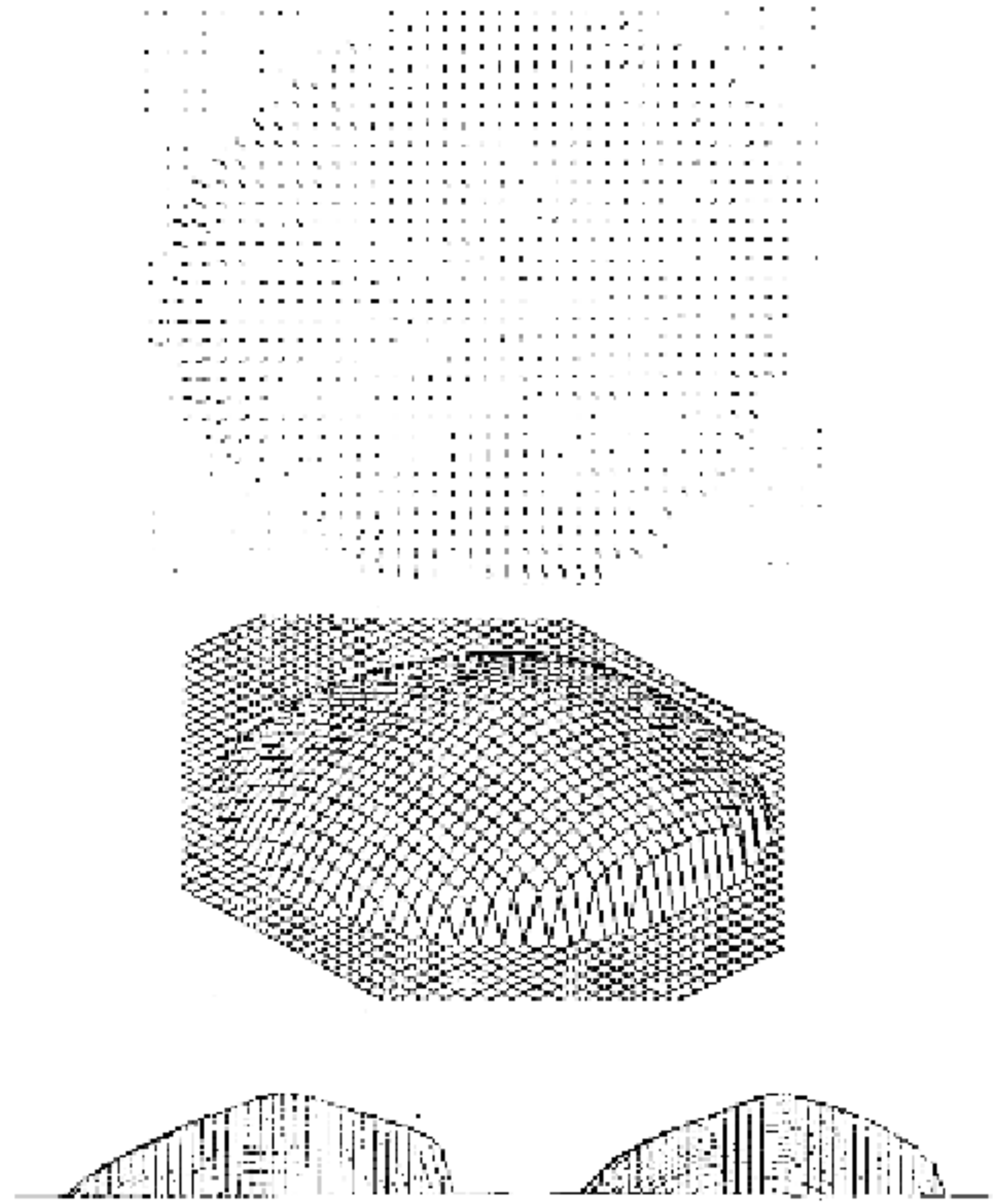


by Ikeuchi and Horn

# Results

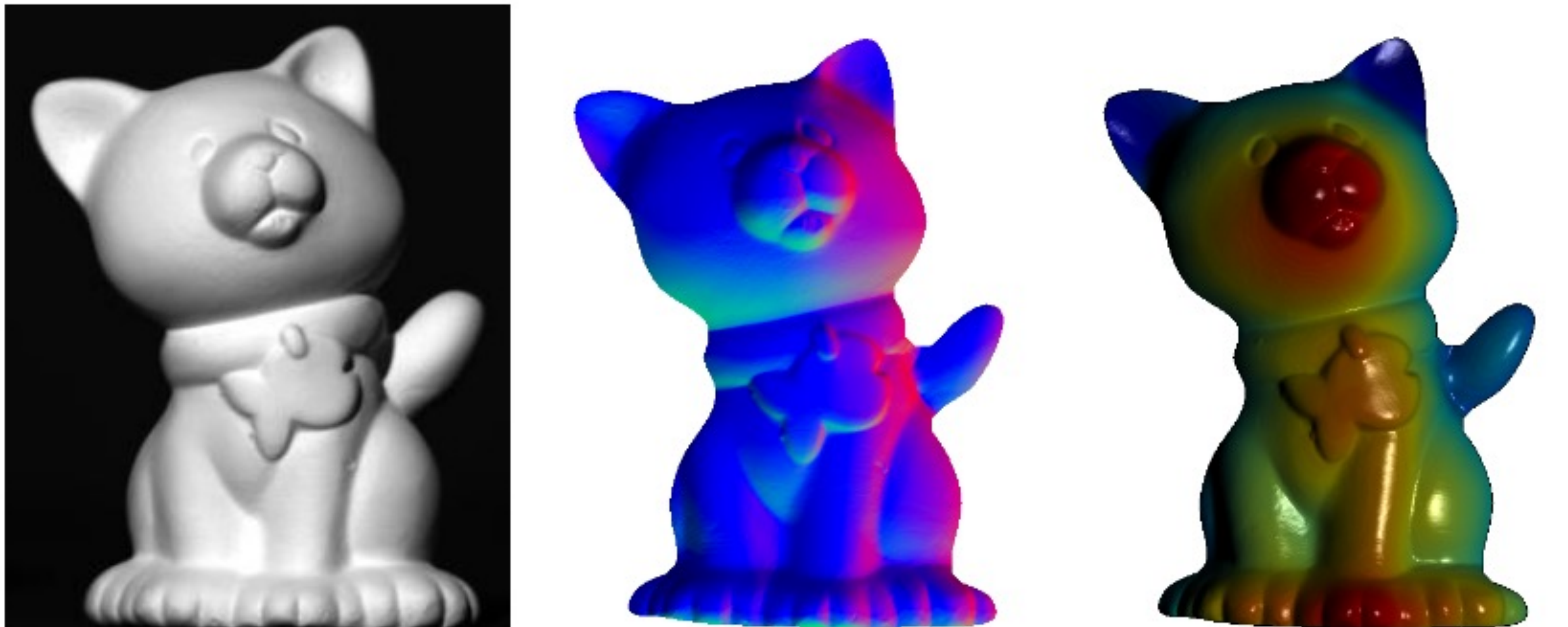


Scanning Electron Microscope image
(inverse intensity)

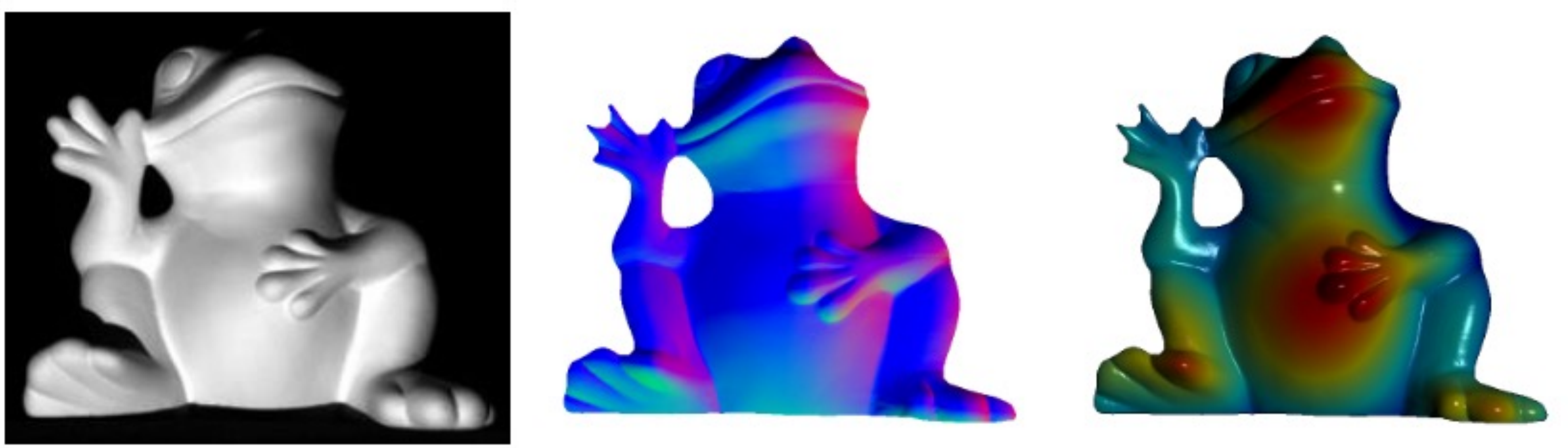


by Ikeuchi and Horn

# More modern results



Resolution: 640 x 500; Re-rendering Error: 0.0075.



Resolution: 590 x 690; Re-rendering Error: 0.0083.

# References

Basic reading:

- Szeliski, Section 2.2.
- Gortler, Chapter 21.
  This book by Steven Gortler has a great *introduction* to radiometry, reflectance, and their use for image formation.

Additional reading:


- Oren and Nayar, “Generalization of the Lambertian model and implications for machine vision,” IJCV 1995.
  The paper introducing the most common model for rough diffuse reflectance.
- Debevec, “Rendering Synthetic Objects into Real Scenes,” SIGGRAPH 1998.
  The paper that introduced the notion of the environment map, the use of chrome spheres for measuring such maps, and the idea that they can be used for easy rendering.
- Lalonde et al., “Estimating the Natural Illumination Conditions from a Single Outdoor Image,” IJCV 2012.
  A paper on estimating outdoors environment maps from just one image.
- Basri and Jacobs, “Lambertian reflectance and linear subspaces,” ICCV 2001.
- Ramamoorthi and Hanrahan, “A signal-processing framework for inverse rendering,” SIGGRAPH 2001.
- Sloan et al., “Precomputed radiance transfer for real-time rendering in dynamic, low-frequency lighting environments,” SIGGRAPH 2002.
  Three papers describing the use of spherical harmonics to model low-frequency illumination, as well as the low-pass filtering effect of Lambertian reflectance on illumination.
- Zhang et al., “Shape-from-shading: a survey,” PAMI 1999.
  A review of perceptual and computational aspects of shape from shading.
- Woodham, “Photometric method for determining surface orientation from multiple images,” Optical Engineering 1980.
  The paper that introduced photometric stereo.
- Yuille and Snow, “Shape and albedo from multiple images using integrability,” CVPR 1997.
- Belhumeur et al., “The bas-relief ambiguity,” IJCV 1999.
- Papadhimitri and Favaro, “A new perspective on uncalibrated photometric stereo,” CVPR 2013.
  Three papers discussing uncalibrated photometric stereo. The first paper shows that, when the lighting directions are not known, by assuming integrability, one can reduce unknowns to the bas-relief ambiguity. The second paper discusses the bas-relief ambiguity in a more general context. The third paper shows that, if instead of an orthographic camera one uses a perspective camera, this is further reduced to just a scale ambiguity.
- Alldrin et al., “Resolving the generalized bas-relief ambiguity by entropy minimization,” CVPR 2007.
  A popular technique for resolving the bas-relief ambiguity in uncalibrated photometric stereo.
- Zickler et al., “Helmholtz stereopsis: Exploiting reciprocity for surface reconstruction,” IJCV 2002.
  A method for photometric stereo reconstruction under arbitrary BRDF.
- Nayar et al., “Shape from interreflections,” IJCV 1991.
- Chandraker et al., “Reflections on the generalized bas-relief ambiguity,” CVPR 2005.
  Two papers discussing how one can perform photometric stereo (calibrated or otherwise) in the presence of strong interreflections.
- Frankot and Chellappa, “A method for enforcing integrability in shape from shading algorithms,” PAMI 1988.
- Agrawal et al., “What is the range of surface reconstructions from a gradient field?,” ECCV 2006.
  Two papers discussing how one can integrate a normal field to reconstruct a surface.